Coincidence arrangements of local observables and uniqueness of the vacuum in QFT

2009 J. Phys. A: Math. Theor. 42 365201

(http://iopscience.iop.org/1751-8121/42/36/365201)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 365201 (22pp) doi:10.1088/1751-8113/42/36/365201

Coincidence arrangements of local observables and uniqueness of the vacuum in QFT

Wojciech Dybalski[1]

Institut für Theoretische Physik, Universität Göttingen, Friedrich-Hund-Platz 1, D-37077 Göttingen, Germany

E-mail: dybalski@ma.tum.de

Received 13 May 2009, in final form 15 May 2009
Published 19 August 2009
Online at stacks.iop.org/JPhysA/42/365201

Abstract

A new phase space criterion, encoding the physically motivated behavior of coincidence arrangements of local observables, is proposed in this work. This condition entails, in particular, uniqueness and purity of the energetically accessible vacuum states. It is shown that the qualitative part of this new criterion is equivalent to a compactness condition proposed in the literature. Its novel quantitative part is verified in massive free field theory.

PACS number: 11.10.Cd
Mathematics Subject Classification: 81T05, 46L40

1. Introduction

Physical properties of vacuum states have been a subject of study since the early days of algebraic quantum field theory [1, 2]. In particular, the problem of convergence of physical states to a vacuum state under large translations attracted much attention. It was considered under the assumptions of complete (Wigner-) particle interpretation [3], sharp mass hyperboloid [4] and asymptotic abelianess in time [5]. As none of these assumptions is expected to hold in all physically relevant models, further investigation of the vacuum structure is warranted. We revisited this subject in recent publications [6, 7]. There we proposed a phase space condition $N_\natural$ which encodes the firm physical principle of additivity of energy over isolated subsystems. It entails the uniqueness of the vacuum states which can be prepared with a finite amount of energy. These vacuum states appear, in particular, as limits of physical states under large timelike translations in Lorentz covariant theories; they can also be approximated by states of increasingly sharp energy–momentum values, in accordance with the uncertainty principle.

[1] Present address: Zentrum Mathematik, Technische Universität München, D-85747 Garching, Germany.

In the present paper we introduce a new phase space condition $C_\flat$, stated below, which is inspired by the fact that all elementary physical states are localized somewhere in space. We show that this new criterion has all the physical consequences listed above and, in addition, entails purity of the vacuum state. A large part of the paper is devoted to the proof that the new criterion holds in a model of massive, non-interacting particles and therefore is consistent with the basic postulates of quantum field theory [8] which we now briefly recall.

The theory is based on a local net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ of von Neumann algebras which are attached to open, bounded regions of spacetime $\mathcal{O} \subset \mathbb{R}^{s+1}$ and act on a Hilbert space $\mathcal{H}$. The global algebra of this net, denoted by $\mathfrak{A}$, is irreducibly represented on this space. Moreover, $\mathcal{H}$ carries a strongly continuous unitary representation of the Poincaré group $\mathbb{R}^{s+1} \rtimes L_+^\uparrow \ni (x, \Lambda) \to U(x, \Lambda)$ which acts geometrically on the net

$$\alpha_{(x,\Lambda)}\mathfrak{A}(\mathcal{O}) = U(x, \Lambda)\mathfrak{A}(\mathcal{O})U(x, \Lambda)^{-1} = \mathfrak{A}(\Lambda\mathcal{O} + x). \tag{1.1}$$

We adopt the usual notation for translated operators $\alpha_x A = A(x)$ and functionals $\varphi_x(A) = \varphi(A(x))$, where $A \in \mathfrak{A}, \varphi \in \mathfrak{A}^*$, and demand that the joint spectrum of the generators $H, P_1, \ldots, P_s$ of translations is contained in the closed forward lightcone $\overline{V}_+$. We denote by P_E the spectral projection of H (the Hamiltonian) on the subspace spanned by vectors of energy lower than E. Finally, we identify the predual of $B(\mathcal{H})$ with the space $\mathcal{T}$ of trace-class operators on $\mathcal{H}$ and denote by $\mathcal{T}_E = P_E\mathcal{T}P_E$ the space of normal functionals of energy bounded by E. The states from $\mathfrak{A}^*$ which belong to the weak* closure of $\mathcal{T}_{E,1}$ for some $E \geqslant 0$ will be called the energetically accessible states. (Here $\mathcal{T}_{E,1}$ denotes the unit ball in the Banach space $\mathcal{T}_E$.)

Important motivation for the present study comes from the refined spectral theory of translation automorphisms [9]. The aim of this theory is to decompose the algebra of observables $\mathfrak{A}$ into subspaces which differ in their behavior under translation automorphisms $\mathbb{R}^{s+1} \ni x \to \alpha_x$. The first step is to identify the pure-point spectrum: suppose that $A \in \mathfrak{A}$ is an eigenvector of the translation automorphisms i.e.

$$\alpha_x A = \mathrm{e}^{\mathrm{i}qx}A, \qquad x \in \mathbb{R}^{s+1} \tag{1.2}$$

for some $q \in \mathbb{R}^{s+1}$. Then A belongs to the center of $\mathfrak{A}$, since by locality

$$\|[A, B]\| = \lim_{|\vec{x}|\to\infty} \|[\alpha_{\vec{x}}A, B]\| = 0, \qquad B \in \mathfrak{A}. \tag{1.3}$$

The irreducibility assumption ensures that the center of $\mathfrak{A}$ consists only of multiples of unity. Hence the pure-point subspace is given by $\mathfrak{A}_{\mathrm{pp}} = \{\beta I | \beta \in \mathbb{C}\}$. Since we do not have a concept of orthogonality in $\mathfrak{A}$, it is not obvious how to choose the complementing continuous subspace $\mathfrak{A}_{\mathrm{c}}$. Suppose, however, that there exists a distinguished projection P_{pp} from $\mathfrak{A}$ onto $\mathfrak{A}_{\mathrm{pp}}$. Then it is natural to set $\mathfrak{A}_{\mathrm{c}} = \ker P_{\mathrm{pp}}$ and there follows the decomposition

$$\mathfrak{A} = \mathfrak{A}_{\mathrm{pp}} \oplus \mathfrak{A}_{\mathrm{c}}. \tag{1.4}$$

Motivated by the ergodic theorem from the setting of groups of unitaries acting on a Hilbert space, we attempt to construct such a projection by averaging the automorphisms $\{\alpha_x\}_{x\in\mathbb{R}^{s+1}}$ over the group $\mathbb{R}^{s+1}$. Thus we consider the approximants

$$P_{\mathrm{pp},L}(A) = \frac{1}{|K_L|}\int_{K_L} \mathrm{d}^{s+1}x\,\alpha_x(A), \qquad A \in \mathfrak{A}, \tag{1.5}$$

where $K_L := \{(x^0, \vec{x}) \in \mathbb{R}^{s+1} | x^0 \in [-L^\varepsilon, L^\varepsilon], |\vec{x}| \leqslant L\}$, $0 < \varepsilon < 1$ and the integrals, defined in the weak sense for any finite $L > 0$, are elements of $B(\mathcal{H})$. By locality, the weak* limit points of the net $\{P_{\mathrm{pp},L}(A)\}_{L>0}$ belong to the commutant of $\mathfrak{A}$. Hence, by the irreducibility assumption, they are multiples of unity. With this information at hand one easily obtains

Theorem 1.1. *Let $\{\omega_L\}_{L>0}$ be a net of states on $\mathfrak{A}$ given by the formula*

$$\omega_L(A) := \omega(P_{pp,L}(A)), \qquad A \in \mathfrak{A}, \tag{1.6}$$

for some state $\omega \in \mathcal{T}$. Let $\omega_0^\gamma \in \mathfrak{A}^*$, $\gamma \in \mathbb{I}$, be the limit points of this net in the weak topology of $\mathfrak{A}^*$ and $\{\omega_{L_\beta} | \beta \in \mathbb{J}^\gamma\}$ the corresponding approximating subnets. Then each such limit point ω_0^γ is a translationally invariant, energetically accessible state which is independent of the state ω.*

It is a simple consequence of the above theorem that for any $\gamma \in \mathbb{I}$ we obtain a projection P_{pp}^γ on the pure-point subspace $\mathfrak{A}_{\mathrm{pp}}$ which has the following form:

$$P_{\mathrm{pp}}^\gamma(A) := w^* - \lim_\beta P_{\mathrm{pp},L_\beta}(A) = \omega_0^\gamma(A) I, \qquad A \in \mathfrak{A}. \tag{1.7}$$

On physical grounds we expect that the translationally invariant, energetically accessible states ω_0^γ are vacuum states which all coincide. In order to establish these facts, we amend the general postulates stated above by some physically motivated phase space conditions: First, we consider the maps $\Pi_E : \mathcal{T}_E \to \mathfrak{A}(\mathcal{O})^*$ given by

$$\Pi_E(\varphi) = \varphi|_{\mathfrak{A}(\mathcal{O})}. \tag{1.8}$$

It was argued by Fredenhagen and Hertel in some unpublished work, quoted in [10], that in physically meaningful theories these maps should satisfy the following condition.

Condition $C_\sharp$. The maps Π_E are compact[2] for any $E \geqslant 0$ and any double cone $\mathcal{O}$.

It is a consequence of this criterion that any energetically accessible and translationally invariant state is a vacuum state, as physically expected. (See e.g. theorem 2.2 (a) of [6]). However, the uniqueness of these vacuum states does not seem to follow from the above assumption. In order to settle this issue, we introduce a strengthened variant of this criterion which is inspired by the behavior of coincidence arrangements of local observables. For this purpose we pick one reference vacuum state $\omega_0 \in \{\omega_0^\gamma | \gamma \in \mathbb{I}\}$, define the corresponding continuous subspace $\mathfrak{A}_{\mathrm{c}} = \ker \omega_0$ and the local continuous subspaces $\mathfrak{A}_{\mathrm{c}}(\mathcal{O}) = \{A \in \mathfrak{A}(\mathcal{O}) | \omega_0(A) = 0\}$. Next, for any double cone $\mathcal{O}$ we introduce the Banach space $(\mathfrak{A}_{\mathrm{c}}(\mathcal{O})^{\times N})^*$ of N-linear forms on $\mathfrak{A}_{\mathrm{c}}(\mathcal{O})$, equipped with the norm

$$\|\psi\| = \sup_{\substack{A_i \in \mathfrak{A}_{\mathrm{c}}(\mathcal{O})_1 \\ i \in \{1,\ldots,N\}}} |\psi(A_1 \times \cdots \times A_N)|. \tag{1.9}$$

In order to control the minimal distance between the regions in which the measurements are performed, we define the set of admissible translates of the region $\mathcal{O}$

$$\Gamma_{N,\delta} = \{\underline{\vec{x}} = (\vec{x}_1, \ldots, \vec{x}_N) \in \mathbb{R}^{Ns} | \forall_{t \in]-\delta,\delta[, i \neq j} \mathcal{O} + \vec{x}_i \sim \mathcal{O} + \vec{x}_j + t\hat{e}_0\}, \tag{1.10}$$

where the symbol $\sim$ indicates spacelike separation and $\hat{e}_0$ is the unit vector in the time direction. For any $\underline{\vec{x}} \in \Gamma_{N,\delta}$ and $\varphi \in \mathcal{T}_E$ we introduce the following elements of $(\mathfrak{A}_{\mathrm{c}}(\mathcal{O})^{\times N})^*$:

$$\varphi_{\underline{\vec{x}}}(A_1 \times \cdots \times A_N) = \varphi(A_1(\vec{x}_1) \cdots A_N(\vec{x}_N)). \tag{1.11}$$

Next, we consider the maps $\Pi_{E,N,\delta} : \mathcal{T}_E \times \Gamma_{N,\delta} \to (\mathfrak{A}_{\mathrm{c}}(\mathcal{O})^{\times N})^*$, given by

$$\Pi_{E,N,\delta}(\varphi, \underline{\vec{x}}) = \varphi_{\underline{\vec{x}}}, \tag{1.12}$$

which are linear in the first argument. Postponing the formal definitions of boundedness and compactness for such maps to section 2, we state a theorem which is at the basis of our investigation.

[2] We adopt the restrictive definition of compactness from [10]: a map is compact if it can be approximated in norm by finite-rank mappings. See section 2 for details.

Theorem 1.2. *A theory satisfies condition* $C_\sharp$ *if and only if the maps* $\Pi_{E,N,\delta}$ *are compact for any* $E \geqslant 0$, $N \in \mathbb{N}$, $\delta > 0$ *and double cone* $\mathcal{O} \subset \mathbb{R}^{s+1}$.

This result, whose proof is given in section 2, opens the possibility to encode the physically expected behavior of coincidence arrangement of detectors into the phase space structure of a theory. We note that any functional from $\mathcal{T}_E$ should describe systems with only a finite number of distinct localization centers. Indeed, in a theory of particles of mass $m > 0$ the maximal number of such centers $N_0(E)$ is given, essentially, by $\frac{E}{m}$. If the number of detectors N is larger than $N_0(E)$, then at least one of them should give no response and the result of the entire coincidence measurement should be zero. We formulate this observation mathematically as a strengthened, quantitative variant of condition $C_\sharp$:

Condition $C_\flat$

(a) The maps $\Pi_{E,N,\delta}$ are compact for any $E \geqslant 0$, $N \in \mathbb{N}$, $\delta > 0$ and double cone $\mathcal{O} \subset \mathbb{R}^{s+1}$.
(b) For any $E \geqslant 0$ there exists $N_0(E) \in \mathbb{N}$ s.t. for any $N > N_0(E)$ the ε-content[3] $\mathcal{N}(\varepsilon)_{E,N,\delta}$ of the map $\Pi_{E,N,\delta}$ satisfies

$$\lim_{\delta\to\infty} \mathcal{N}(\varepsilon)_{E,N,\delta} = 1 \tag{1.13}$$

for any $\varepsilon > 0$.

It is our main result that the reference state ω_0, which enters into the definition of the maps $\Pi_{E,N,\delta}$, is the unique, energetically accessible vacuum state in theories complying with condition $C_\flat$. Thus, it defines a distinguished projection $P_{\mathrm{pp}}(\cdot) = \omega_0(\cdot)I$ on $\mathfrak{A}_{\mathrm{pp}}$ which fixes decomposition (1.4).

Our paper is organized as follows: in section 2, we show that the qualitative part (a) of condition $C_\flat$ is equivalent to condition $C_\sharp$ (i.e. we prove theorem 1.2 stated above). In section 3, we study physical consequences of condition $C_\flat$ which include uniqueness and purity of the energetically accessible vacuum state as well as various preparation procedures for this state. In section 4, we verify the quantitative part (b) of the new criterion in massive free field theory. The more technical part of this discussion is postponed to the appendices. The paper closes with brief conclusions.

The results presented in this paper were included in the PhD Thesis of the author [9] completed at the University of Göttingen.

2. Equivalence of conditions $C_\flat$(a) and $C_\sharp$

In this section, we show that condition $C_\flat$(a) is equivalent to the existing condition $C_\sharp$ i.e. we prove theorem 1.2 stated in section 1.

First, we specify the notions of compactness which are used in the formulation of conditions $C_\sharp$ and $C_\flat$: let V and W be Banach spaces and let $\mathcal{L}(V,W)$ denote the space of linear maps from V to W equipped with the standard norm. Let $\mathcal{F}(V,W)$ denote the subspace of finite-rank mappings. More precisely, any $F \in \mathcal{F}(V,W)$ is of the form $F = \sum_{i=1}^n \tau_i S_i$, where $\tau_i \in W$ and $S_i \in V^*$. We say that a map $\Pi \in \mathcal{L}(V,W)$ is compact, if it belongs to the closure of $\mathcal{F}(V,W)$ in the norm topology of $\mathcal{L}(V,W)$. This concept is used in condition $C_\sharp$. To formulate a notion of compactness which is adequate for condition $C_\flat$, we need a more

[3] Let V, W be Banach spaces and let Γ be some set. Then the ε-content of a map $\Pi : V \times \Gamma \to W$ is the maximal natural number $\mathcal{N}(\varepsilon)$ for which there exist elements $(v_1, x_1), \ldots, (v_{\mathcal{N}(\varepsilon)}, x_{\mathcal{N}(\varepsilon)}) \in V_1 \times \Gamma$ s.t. $\|\Pi(v_i, x_i) - \Pi(v_j, x_j)\| > \varepsilon$ for $i \neq j$.

general framework: let Γ be a set and let $\mathcal{L}(V \times \Gamma, W)$ be the space of maps from $V \times \Gamma$ to W, linear in the first argument, which are bounded in the norm

$$\|\Pi\| = \sup_{\substack{v\in V_1 \\ x\in\Gamma}} \|\Pi(v, x)\|. \tag{2.1}$$

The subspace of finite-rank maps $\mathcal{F}(V \times \Gamma, W)$ contains all the maps of the form $F = \sum_{i=1}^{n} \tau_i S_i$, where $\tau_i \in W, S_i \in \mathcal{L}(V \times \Gamma, \mathbb{C})$. We say that a map $\Pi \in \mathcal{L}(V \times \Gamma, W)$ is compact, if it belongs to the closure of $\mathcal{F}(V \times \Gamma, W)$ in the norm topology of $\mathcal{L}(V \times \Gamma, W)$.

Proof. $C_\flat(\mathrm{a}) \Rightarrow C_\sharp$: setting $N = 1$ in condition $C_\flat(\mathrm{a})$, we obtain, for any $\varepsilon > 0$, a finite-rank map $F \in \mathcal{F}(\mathcal{T}_E \times \mathbb{R}^s, \mathfrak{A}_\mathrm{c}(\mathcal{O})^*)$ s.t.

$$\sup_{\substack{(\varphi,\vec{x})\in\mathcal{T}_{E,1}\times\mathbb{R}^s \\ A\in\mathfrak{A}_\mathrm{c}(\mathcal{O})_1}} |\Pi_{E,1,\delta}(\varphi, \vec{x})(A) - F(\varphi, \vec{x})(A)| \leqslant \varepsilon. \tag{2.2}$$

Noting that $\Pi_{E,1,\delta}(\varphi, \vec{x}) = \Pi_E(\varphi_{\vec{x}})|_{\mathfrak{A}_\mathrm{c}(\mathcal{O})}$ and making use of the fact that $\frac{1}{2}(A - \omega_0(A)I) \in \mathfrak{A}_\mathrm{c}(\mathcal{O})_1$ for any $A \in \mathfrak{A}(\mathcal{O})_1$, we obtain

$$\sup_{\substack{\varphi\in\mathcal{T}_{E,1} \\ A\in\mathfrak{A}(\mathcal{O})_1}} |\Pi_E(\varphi)(A) - \varphi(I)\omega_0(A) - F(\varphi, 0)(A - \omega_0(A)I)| \leqslant 2\varepsilon. \tag{2.3}$$

Thus, we can approximate the map Π_E in norm with finite-rank mappings up to an arbitrary accuracy, i.e. this map is compact. $\square$

The opposite implication is more interesting. It says that the restriction imposed by condition $C_\sharp$ on the number of states which can be distinguished by measurements with singly localized detectors limits also the number of states which can be discriminated by coincidence arrangements of such detectors. We start our analysis from the observation that the spatial distance between the detectors suppresses the energy transfer between them. The proof of the following lemma relies on methods from [11].

Lemma 2.1. *Let $\delta > 0, \beta > 0$. Define the function $g :]-\pi, \pi] \to \mathbb{C}$ as follows*

$$g(\phi) = \frac{\beta}{\pi} \ln \left| \cot \frac{\phi + \gamma}{2} \cot \frac{\phi - \gamma}{2} \right|, \tag{2.4}$$

where $\gamma = 2 \arctan \mathrm{e}^{-\frac{\pi\delta}{2\beta}}$. Then, for any pair of bounded operators A, B satisfying $[A(t), B] = 0$ for $|t| < \delta$ and any functional $\varphi \in \mathrm{e}^{-\beta H}\mathcal{T}\mathrm{e}^{-\beta H}$, there holds the identity

$$\varphi(AB) = \varphi([A, \mathring{B}_\beta]_+) + \varphi(A\mathrm{e}^{-\beta H} B_\beta \mathrm{e}^{\beta H}) + \varphi(\mathrm{e}^{\beta H} B_\beta \mathrm{e}^{-\beta H} A), \tag{2.5}$$

where $[\cdot, \cdot]_+$ denotes the anti-commutator and we made use of the fact that $\varphi(\mathrm{e}^{\beta H}\,\cdot\,), \varphi(\,\cdot\,\mathrm{e}^{\beta H})$ are elements of $\mathcal{T}$. Here $\mathring{B}_\beta$ and B_β are elements of $B(\mathcal{H})$ given by the (weak) integrals

$$\mathring{B}_\beta = \frac{1}{2\pi} \int_0^\gamma \mathrm{d}\phi\, B(g(\phi)) + \frac{1}{2\pi} \int_{\pi-\gamma}^{\pi} \mathrm{d}\phi\, B(g(\phi)), \tag{2.6}$$

$$B_\beta = \frac{1}{2\pi} \int_\gamma^{\pi-\gamma} \mathrm{d}\phi\, B(g(\phi)), \tag{2.7}$$

where $B(g(\phi)) = \mathrm{e}^{\mathrm{i}g(\phi)H} B \mathrm{e}^{-\mathrm{i}g(\phi)H}$.

Proof. It suffices to prove the statement for functionals of the form $\varphi(\cdot) = (\Psi_1| \cdot \Psi_2)$, where Ψ_1 and Ψ_2 are vectors from the domain of $\mathrm{e}^{\beta H}$. For $\delta > 0$ and $\beta > 0$ we define the set

$$G_{\beta,\delta} = \{z \in \mathbb{C} | |\mathrm{Im} z| < \beta\} \backslash \{z | \mathrm{Im} z = 0, |\mathrm{Re} z| \geqslant \delta\} \tag{2.8}$$

and introduce the following function, analytic on $G_{\beta,\delta}$ and continuous at its boundary

$$h(z) = \begin{cases} (\Psi_1|A\mathrm{e}^{\mathrm{i}zH}B\mathrm{e}^{-\mathrm{i}zH}\Psi_2) & \text{for } 0 < \mathrm{Im}z < \beta \\ (\Psi_1|\mathrm{e}^{\mathrm{i}zH}B\mathrm{e}^{-\mathrm{i}zH}A\Psi_2) & \text{for } -\beta < \mathrm{Im}z < 0 \\ (\Psi_1|AB(z)\Psi_2) = (\Psi_1|B(z)A\Psi_2) & \text{for } \mathrm{Im}z = 0 \text{ and } |\mathrm{Re}z| < \delta. \end{cases} \tag{2.9}$$

We make use of the following conformal mapping from the unit disc $\{w||w| < 1\}$ to $G_{\beta,\delta}$ [11]

$$z(w) = \frac{\beta}{\pi}\left\{\ln\frac{1+w\mathrm{e}^{\mathrm{i}\gamma}}{1-w\mathrm{e}^{\mathrm{i}\gamma}} - \ln\frac{1-w\mathrm{e}^{-\mathrm{i}\gamma}}{1+w\mathrm{e}^{-\mathrm{i}\gamma}}\right\}. \tag{2.10}$$

Setting $w = r\mathrm{e}^{\mathrm{i}\phi}$, $0 < r < 1$, we obtain from the Cauchy formula

$$h(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\phi h(z(r\mathrm{e}^{\mathrm{i}\phi})). \tag{2.11}$$

Since $h(z)$ satisfies the following bound on the closure of $G_{\beta,\delta}$:

$$|h(z)| \leqslant \|A\|\|B\|\|\mathrm{e}^{\beta H}\Psi_1\|\|\mathrm{e}^{\beta H}\Psi_2\|, \tag{2.12}$$

we can, by the dominated convergence theorem, extend the path of integration in relation (2.11) to the circle $r = 1$. In this limit we have [11]

$$\mathrm{Re}\, z(\mathrm{e}^{\mathrm{i}\phi}) = g(\phi), \tag{2.13}$$

$$\mathrm{Im}\, z(\mathrm{e}^{\mathrm{i}\phi}) = \begin{cases} 0 & \text{if } |\phi| < \gamma \text{ or } \pi - \phi < \gamma \text{ or } \pi + \phi < \gamma \\ \beta & \text{if } \gamma < \phi < \pi - \gamma \\ -\beta & \text{if } \gamma < -\phi < \pi - \gamma. \end{cases} \tag{2.14}$$

Consequently, we obtain from (2.11)

$$\begin{aligned}(\Psi_1|AB\Psi_2) = {} & \frac{1}{2\pi}\int_0^{\gamma} \mathrm{d}\phi (\Psi_1|[A, B(g(\phi))]_+\Psi_2) + \frac{1}{2\pi}\int_{\pi-\gamma}^{\pi} \mathrm{d}\phi (\Psi_1|[A, B(g(\phi))]_+\Psi_2) \\ & + \frac{1}{2\pi}\int_{\gamma}^{\pi-\gamma} \mathrm{d}\phi \big((\Psi_1|A\mathrm{e}^{-\beta H}B(g(\phi))\mathrm{e}^{\beta H}\Psi_2) + (\mathrm{e}^{\beta H}\Psi_1|B(g(\phi))\mathrm{e}^{-\beta H}A\Psi_2)\big),\end{aligned} \tag{2.15}$$

which concludes the proof. □

In order to complete the proof of theorem 1.2, we have to proceed from the sharp energy bounds assumed in condition $C_\sharp$ to the exponential energy damping which is established in lemma 2.1. Making use of the fact that $\mathcal{T}_E^* = P_E B(\mathcal{H}) P_E$, condition $C_\sharp$ can be restated as a requirement that the maps $\widehat{\Xi}_E : \mathfrak{A}(\mathcal{O}) \to B(\mathcal{H})$, given by $\widehat{\Xi}_E(A) = P_E A P_E$, are compact for any $E \geqslant 0$ and any double cone $\mathcal{O}$. (See [10] for a similar discussion). With the help of the estimate

$$\|\mathrm{e}^{-\beta H}A\mathrm{e}^{-\beta H} - \mathrm{e}^{-\beta H}P_E A P_E \mathrm{e}^{-\beta H}\| \leqslant 2\|A\|\mathrm{e}^{-\beta E}, \tag{2.16}$$

one also concludes that the maps Ξ_β and Ξ_{β_1,β_2} from $\mathcal{L}(\mathfrak{A}(\mathcal{O}), B(\mathcal{H}))$, defined as

$$\Xi_\beta(A) = \mathrm{e}^{-\beta H}A\mathrm{e}^{-\beta H}, \tag{2.17}$$

$$\Xi_{\beta_1,\beta_2}(A) = \mathrm{e}^{-\beta_1 H}A_{\beta_2}\mathrm{e}^{-\beta_1 H}, \tag{2.18}$$

are compact for any $\beta, \beta_1, \beta_2 > 0$ and any double cone $\mathcal{O}$. A_{β_2} is given by definition (2.7).

Proof. $C_\sharp \Rightarrow C_\flat$(a): for any $\beta > 0$ we introduce the auxiliary maps $\widehat{\Pi}_{\beta,N,\delta} \in \mathcal{L}(\mathcal{T} \times \Gamma_{N,\delta}, (\mathfrak{A}_{\mathrm{c}}(\mathcal{O})^{\times N})^*)$ given by

$$\widehat{\Pi}_{\beta,N,\delta}(\varphi, \underline{\vec{x}})(A_1 \times \cdots \times A_N) = \varphi(\mathrm{e}^{-(N+\frac{1}{2})\beta H} A_1(\vec{x}_1) \cdots A_N(\vec{x}_N)\mathrm{e}^{-(N+\frac{1}{2})\beta H}). \tag{2.19}$$

They are related to the maps $\Pi_{E,N,\delta} \in \mathcal{L}(\mathcal{T}_E \times \Gamma_{N,\delta}, (\mathfrak{A}_{\mathrm{c}}(\mathcal{O})^{\times N})^*)$ by the following identity, valid for any $\varphi \in \mathcal{T}_E$

$$\Pi_{E,N,\delta}(\varphi, \underline{\vec{x}}) = \widehat{\Pi}_{\beta,N,\delta}(\mathrm{e}^{(N+\frac{1}{2})\beta H}\varphi \mathrm{e}^{(N+\frac{1}{2})\beta H}, \underline{\vec{x}}). \tag{2.20}$$

In order to prove the compactness of the maps $\Pi_{E,N,\delta}$, it suffices to verify that the family of mappings $\{\widehat{\Pi}_{\beta,N,\delta}\}_{\beta>0}$ is *asymptotically compact* in the following sense: there exists a family of finite-rank maps $\widehat{F}_{\beta,N,\delta} \in \mathcal{F}(\mathcal{T} \times \Gamma_{N,\delta}, (\mathfrak{A}_{\mathrm{c}}(\mathcal{O})^{\times N})^*)$ s.t.

$$\lim_{\beta\to 0} \|\widehat{\Pi}_{\beta,N,\delta} - \widehat{F}_{\beta,N,\delta}\| = 0. \tag{2.21}$$

If this property holds, then, by identity (2.20), the maps $\Pi_{E,N,\delta}$ can be approximated in norm as $\beta \to 0$ by the finite-rank maps $F_{\beta,N,\delta} \in \mathcal{L}(\mathcal{T}_E \times \Gamma_{N,\delta}, (\mathfrak{A}_{\mathrm{c}}(\mathcal{O})^{\times N})^*)$ defined as

$$F_{\beta,N,\delta}(\varphi, \underline{\vec{x}}) = \widehat{F}_{\beta,N,\delta}(\mathrm{e}^{(N+\frac{1}{2})\beta H}\varphi \mathrm{e}^{(N+\frac{1}{2})\beta H}, \underline{\vec{x}}). \tag{2.22}$$

We establish property (2.21) by induction in N: for $N = 1$ the statement follows from compactness of the map $\Xi_{\frac{3}{2}\beta}$ given by (2.17). Next, we assume that the family $\{\widehat{\Pi}_{\beta,N-1,\delta}\}_{\beta>0}$ is asymptotically compact and prove that $\{\widehat{\Pi}_{\beta,N,\delta}\}_{\beta>0}$ also has this property. For this purpose, we pick $\varphi \in \mathcal{T}_1, A_1, \ldots, A_N \in \mathfrak{A}_{\mathrm{c}}(\mathcal{O})_1$ and $\underline{\vec{x}} \in \Gamma_{N,\delta}$. Then $A_1(\vec{x}_1)\ldots A_{N-1}(\vec{x}_{N-1})$ and $A_N(\vec{x}_N)$ satisfy the assumptions of lemma 2.1 and we obtain

$$\begin{aligned}
&\widehat{\Pi}_{\beta,N,\delta}(\varphi, \underline{\vec{x}})(A_1 \times \cdots \times A_N) \\
&\quad = \varphi(\mathrm{e}^{-(N+\frac{1}{2})\beta H}[A_1(\vec{x}_1)\ldots A_{N-1}(\vec{x}_{N-1}), \mathring{A}_{N,N\beta}(\vec{x}_N)]_+ \mathrm{e}^{-(N+\frac{1}{2})\beta H}) \\
&\qquad + \widehat{\Pi}_{\beta,N-1,\delta}\big(\{\Xi_{\frac{1}{2}\beta,N\beta}(A_N)(\vec{x}_N)\varphi \mathrm{e}^{-\beta H}\}, \vec{x}_1, \ldots, \vec{x}_{N-1}\big)(A_1 \times \cdots \times A_{N-1}) \\
&\qquad + \widehat{\Pi}_{\beta,N-1,\delta}(\{\mathrm{e}^{-\beta H}\varphi \Xi_{\frac{1}{2}\beta,N\beta}(A_N)(\vec{x}_N)\}, \vec{x}_1, \ldots, \vec{x}_{N-1})(A_1 \times \cdots \times A_{N-1}).
\end{aligned} \tag{2.23}$$

The first term on the rhs of (2.23) satisfies

$$|\varphi(\mathrm{e}^{-(N+\frac{1}{2})\beta H}[A_1(\vec{x}_1)\cdots A_{N-1}(\vec{x}_{N-1}), \mathring{A}_{N,N\beta}(\vec{x}_N)]_+ \mathrm{e}^{-(N+\frac{1}{2})\beta H})| \leqslant \frac{2\gamma(N\beta)}{\pi}, \tag{2.24}$$

where we made use of definition (2.6). We recall from the statement of lemma 2.1 that $\gamma(N\beta) \to 0$ with $\beta \to 0$. To treat the remaining terms, we make use of the induction hypothesis: it assures that there exist finite-rank mappings $\widehat{F}_{\beta,N-1,\delta} \in \mathcal{F}(\mathcal{T} \times \Gamma_{N-1,\delta}, (\mathfrak{A}_{\mathrm{c}}(\mathcal{O})^{\times N-1})^*)$ s.t.

$$\lim_{\beta\to 0} \|\widehat{\Pi}_{\beta,N-1,\delta} - \widehat{F}_{\beta,N-1,\delta}\| = 0. \tag{2.25}$$

Next, making use of the compactness of the maps $\Xi_{\frac{1}{2}\beta,N\beta} \in \mathcal{L}(\mathfrak{A}(\mathcal{O}), B(\mathcal{H}))$, given by formula (2.18), we can find a family of finite-rank mappings $F_\beta \in \mathcal{F}(\mathfrak{A}(\mathcal{O}), B(\mathcal{H}))$ s.t.

$$\|\Xi_{\frac{1}{2}\beta,N\beta} - F_\beta\| \leqslant \frac{\beta}{1 + \|\widehat{F}_{\beta,N-1,\delta}\|}, \tag{2.26}$$

for any $\beta > 0$. Now the second term on the rhs of (2.23) can be rewritten as follows:

$$\begin{aligned}
&\widehat{\Pi}_{\beta,N-1,\delta}(\{\Xi_{\frac{1}{2}\beta,N\beta}(A_N)(\vec{x}_N)\varphi \mathrm{e}^{-\beta H}\}, \vec{x}_1, \ldots, \vec{x}_{N-1}) \\
&\quad = (\widehat{\Pi}_{\beta,N-1,\delta} - \widehat{F}_{\beta,N-1,\delta})(\{\Xi_{\frac{1}{2}\beta,N\beta}(A_N)(\vec{x}_N)\varphi \mathrm{e}^{-\beta H}\}, \vec{x}_1, \ldots, \vec{x}_{N-1}) \\
&\qquad + \widehat{F}_{\beta,N-1,\delta}(\{(\Xi_{\frac{1}{2}\beta,N\beta}(A_N) - F_\beta(A_N))(\vec{x}_N)\varphi \mathrm{e}^{-\beta H}\}, \vec{x}_1, \ldots, \vec{x}_{N-1}) \\
&\qquad + \widehat{F}_{\beta,N-1,\delta}(\{F_\beta(A_N)(\vec{x}_N)\varphi \mathrm{e}^{-\beta H}\}, \vec{x}_1, \ldots, \vec{x}_{N-1}).
\end{aligned} \tag{2.27}$$

We obtain from relations (2.25) and (2.26) that the first two terms on the rhs of equation (2.27) tend to zero with $\beta \to 0$ in the norm topology of $(\mathfrak{A}_c(\mathcal{O})^{\times N-1})^*$, uniformly in $\varphi \in \mathcal{T}_1$, $A_N \in \mathfrak{A}_c(\mathcal{O})_1$ and $\underline{\vec{x}} \in \Gamma_{N,\delta}$. The last term on the rhs of relation (2.27) coincides with the finite-rank map $\widehat{F}^{(1)}_{\beta,N,\delta} \in \mathcal{F}(\mathcal{T} \times \Gamma_{N,\delta}, (\mathfrak{A}_c(\mathcal{O})^{\times N})^*)$, given by

$$\widehat{F}^{(1)}_{\beta,N,\delta}(\varphi, \vec{x})(A_1 \times \cdots \times A_N) \\ = \widehat{F}_{\beta,N-1,\delta}(\{F_\beta(A_N)(\vec{x}_N)\varphi \mathrm{e}^{-\beta H}\}, \vec{x}_1, \ldots, \vec{x}_{N-1})(A_1 \times \cdots \times A_{N-1}). \quad (2.28)$$

The last term on the rhs of (2.23) can be analogously approximated by the maps $\widehat{F}^{(2)}_{\beta,N,\delta} \in \mathcal{F}(\mathcal{T} \times \Gamma_{N,\delta}, (\mathfrak{A}_c(\mathcal{O})^{\times N})^*)$ defined as

$$\widehat{F}^{(2)}_{\beta,N,\delta}(\varphi, \vec{x})(A_1 \times \cdots \times A_N) \\ = \widehat{F}_{\beta,N-1,\delta}(\{\mathrm{e}^{-\beta H}\varphi F_\beta(A_N)(\vec{x}_N)\}, \vec{x}_1, \ldots, \vec{x}_{N-1})(A_1 \times \cdots \times A_{N-1}). \quad (2.29)$$

Summing up, we obtain from (2.23) and (2.24) that

$$\lim_{\beta \to 0} \|\widehat{\Pi}_{\beta,N,\delta} - \widehat{F}^{(1)}_{\beta,N,\delta} - \widehat{F}^{(2)}_{\beta,N,\delta}\| = 0, \quad (2.30)$$

which concludes the inductive argument and the proof of theorem 1.2. □

3. Condition $C_\flat$ and the vacuum structure

In this section we show that the vacuum state ω_0, which entered into the formulation of condition $C_\flat$, is pure and that it is the only energetically accessible vacuum state. Similarly as in [6], there follows relaxation of physical states to ω_0 under large timelike translations and appearance of this vacuum state as a limit of states of increasingly sharp energy–momentum values.

We recall that the ε-contents of the maps $\Pi_{E,N,\delta}$, which entered into the formulation of condition $C_\flat$, have direct physical interpretation: they restrict the number of different measurement results which may occur in coincidence arrangements of local operators from $\mathfrak{A}_c(\mathcal{O})$. However, for many applications it is more convenient to work with the norms of the maps $\Pi_{E,N,\delta}$. The link is provided by the following lemma.

Lemma 3.1. *Let V and W be Banach spaces and let $\{\Gamma_\delta\}_{\delta>0}$ be a family of sets ordered by inclusion i.e. $\Gamma_{\delta_1} \subset \Gamma_{\delta_2}$ for $\delta_1 \geqslant \delta_2$. Let $\{\Pi_\delta\}_{\delta>0}$ be a family of compact maps from $\mathcal{L}(V \times \Gamma_\delta, W)$ and let $\mathcal{N}(\varepsilon)_\delta$ be the respective ε-contents. Then there holds $\lim_{\delta\to\infty} \mathcal{N}(\varepsilon)_\delta = 1$ for any $\varepsilon > 0$ if and only if $\lim_{\delta\to\infty} \|\Pi_\delta\| = 0$.*

Proof. First, suppose that $\lim_{\delta\to\infty} \mathcal{N}(\varepsilon)_\delta = 1$ for any $\varepsilon > 0$. Since the ε-content takes only integer values, for any $\varepsilon > 0$ we can choose δ_ε s.t. $\mathcal{N}(\varepsilon)_\delta = 1$ for $\delta \geqslant \delta_\varepsilon$. Then, by definition of the ε-content, there holds for any $\delta \geqslant \delta_\varepsilon$

$$\|\Pi_\delta\| \leqslant \varepsilon, \quad (3.1)$$

which entails $\lim_{\delta\to\infty} \|\Pi_\delta\| = 0$.

To prove the opposite implication, we proceed by contradiction: we recall that the ε-content of a compact map is finite for any $\varepsilon > 0$. Next, we note that for any fixed $\varepsilon > 0$ the function $\delta \to \mathcal{N}(\varepsilon)_\delta$ is decreasing and bounded from below by one, so there exists $\lim_{\delta\to\infty} \mathcal{N}(\varepsilon)_\delta$. Suppose that this limit is strictly larger than one. Then, by definition of the ε-content, there exist nets $(\varphi_1^{(\delta)}, \vec{x}_1^{(\delta)})$ and $(\varphi_2^{(\delta)}, \vec{x}_2^{(\delta)})$ in $V_1 \times \Gamma_\delta$ s.t.

$$\big\|\Pi_\delta\big(\varphi_1^{(\delta)}, \vec{x}_1^{(\delta)}\big) - \Pi_\delta\big(\varphi_2^{(\delta)}, \vec{x}_2^{(\delta)}\big)\big\| > \varepsilon \quad (3.2)$$

for any $\delta > 0$. However, this inequality contradicts the assumption that the norms of the maps Π_δ tend to zero with $\delta \to \infty$. □

With the help of the above lemma we reformulate condition $C_\flat$(b) as follows: for any $E \geqslant 0$ there exists such natural number $N_0(E)$ that for any $N > N_0(E)$

$$\lim_{\delta\to\infty} \sup_{\substack{A_i \in \mathfrak{A}_{\mathrm{c}}(\mathcal{O})_1 \\ i\in\{1,\ldots,N\} \\ \underline{\vec{x}}\in\Gamma_{N,\delta}}} \|P_E A_1(\vec{x}_1)\cdots A_N(\vec{x}_N)P_E\| = 0. \tag{3.3}$$

We use this relation in the following key lemma.

Lemma 3.2. *Suppose that condition $C_\flat$ holds. Then, for any $E \geqslant 0$, double cone $\mathcal{O}$, and a sequence $\{\delta(n)\}_1^\infty$ s.t. $\delta(n)\underset{n\to\infty}{\longrightarrow}\infty$, the following assertions hold true:*

(a) For any family of points $\{\vec{x}_i^{(n)}\}_1^n \in \Gamma_{n,\delta(n)}$ there holds

$$\lim_{n\to\infty} \sup_{\substack{\varphi\in\mathcal{T}_{E,1} \\ A\in\mathfrak{A}_{\mathrm{c}}(\mathcal{O})_1}} \left|\frac{1}{n}\sum_{i=1}^n \varphi\big(A\big(\vec{x}_i^{(n)}\big)\big)\right| = 0. \tag{3.4}$$

(b) For any unit vector $\hat{e} \in \mathbb{R}^s$, sequence $\{\lambda^{(n)}\}_1^\infty \in \mathbb{R}_+$ and a family of points $\{\vec{x}_i^{(n)}\}_1^n$, s.t. $\{\vec{x}_i^{(n)}\}_1^n \cup \{\vec{x}_i^{(n)} + \lambda^{(n)}\hat{e}\}_1^n \in \Gamma_{2n,\delta(n)}$, there holds

$$\lim_{n\to\infty} \sup_{\substack{\varphi\in\mathcal{T}_{E,1} \\ A,B\in\mathfrak{A}_{\mathrm{c}}(\mathcal{O})_1}} \left|\frac{1}{n}\sum_{i=1}^n \varphi\big(A\big(\vec{x}_i^{(n)}\big)B\big(\vec{x}_i^{(n)} + \lambda^{(n)}\hat{e}\big)\big)\right| = 0. \tag{3.5}$$

Proof. It is a well-known fact that any normal, self-adjoint functional on a von Neumann algebra can be expressed as a difference of two normal, positive functionals which are mutually orthogonal [12]. It follows that any $\varphi \in \mathcal{T}_{E,1}$ can be decomposed as

$$\varphi = \varphi_{\mathrm{Re}}^+ - \varphi_{\mathrm{Re}}^- + i(\varphi_{\mathrm{Im}}^+ - \varphi_{\mathrm{Im}}^-), \tag{3.6}$$

where $\varphi_{\mathrm{Re}}^\pm, \varphi_{\mathrm{Im}}^\pm$ are positive functionals from $\mathcal{T}_{E,1}$. Therefore, it suffices to prove relations (3.4) and (3.5) for the set $\mathcal{T}_{E,1}^+$ of positive functionals from $\mathcal{T}_{E,1}$. By a similar argument, it suffices to consider self-adjoint operators $A, B \in \mathfrak{A}_{\mathrm{c}}(\mathcal{O})$ in both statements.

We choose some positive functional $\varphi \in \mathcal{T}_{E,1}^+$, pick $m \in \mathbb{N}$ s.t. $N = 2^m$ is sufficiently large to ensure that (3.3) holds. To prove (a), we define operators $Q_n = \frac{1}{n}\sum_{i=1}^n A(\vec{x}_i^{(n)})$, $n \in \mathbb{N}$, where $A \in \mathfrak{A}_{\mathrm{c}}(\mathcal{O})$ is self-adjoint, assume that $n \geqslant N$ and compute

$$\begin{aligned}
|\varphi(Q_n)|^N &\leqslant \varphi\big(Q_n^N\big) = \frac{1}{n^N}\sum_{i_1,\ldots,i_N} \varphi\big(A\big(\vec{x}_{i_1}^{(n)}\big)\cdots A\big(\vec{x}_{i_N}^{(n)}\big)\big) \\
&= \frac{1}{n^N}\sum_{\substack{i_1,\ldots,i_N \\ \forall_{k\neq l}\, i_k\neq i_l}} \varphi\big(A\big(\vec{x}_{i_1}^{(n)}\big)\cdots A\big(\vec{x}_{i_N}^{(n)}\big)\big) \\
&\quad + \frac{1}{n^N}\sum_{\substack{i_1,\ldots,i_N \\ \exists_{k\neq l}\, \text{s.t.}\, i_k = i_l}} \varphi\big(A\big(\vec{x}_{i_1}^{(n)}\big)\cdots A\big(\vec{x}_{i_N}^{(n)}\big)\big) \\
&\leqslant \frac{1}{n^N}\sum_{\substack{i_1,\ldots,i_N \\ \forall_{k\neq l}\, i_k\neq i_l}} \|P_E A\big(\vec{x}_{i_1}^{(n)}\big)\cdots A\big(\vec{x}_{i_N}^{(n)}\big)P_E\| + \frac{1}{n^N}\sum_{\substack{i_1,\ldots,i_N \\ \exists_{k\neq l}\, \text{s.t.}\, i_k = i_l}} \|A\|^N.
\end{aligned} \tag{3.7}$$

In the first step above we applied the Cauchy–Schwarz inequality and in the third step we extracted from the resulting sum the terms in which all the operators are mutually spacelike separated. Clearly, there are $\binom{n}{N}N! \leqslant n^N$ such terms. Therefore, the remainder (the last sum above) consists of

$$n^N - \binom{n}{N}N! \leqslant c_N n^{N-1} \tag{3.8}$$

terms. There follows the estimate

$$|\varphi(Q_n)|^N \leqslant \sup_{\substack{A\in\mathfrak{A}_{\mathrm{c}}(\mathcal{O})_1 \\ (\vec{x}_1,\ldots,\vec{x}_N)\in\Gamma_{N,\delta(n)}}} \|P_E A(\vec{x}_1)\cdots A(\vec{x}_N)P_E\| + \frac{c_N}{n}\|A\|^N, \tag{3.9}$$

whose rhs tends to zero with $n \to \infty$ by (3.3), uniformly in $\varphi \in \mathcal{T}^+_{E,1}$, what concludes the proof of (3.4).

In order to prove (b), we proceed similarly: let $\hat{Q}_n = \frac{1}{n}\sum_{i=1}^n A\big(\vec{x}_i^{(n)}\big)B\big(\vec{x}_i^{(n)} + \lambda^{(n)}\hat{e}\big)$, where $A, B \in \mathfrak{A}_{\mathrm{c}}(\mathcal{O})$ are self-adjoint. Then, for $n \geqslant N$, we obtain

$$\begin{aligned}
|\varphi(\hat{Q}_n)|^N &\leqslant \varphi(\hat{Q}_n^N) \\
&= \frac{1}{n^N}\sum_{i_1,\ldots,i_N} \varphi\big(A\big(\vec{x}_{i_1}^{(n)}\big)B\big(\vec{x}_{i_1}^{(n)} + \lambda^{(n)}\hat{e}\big)\cdots A\big(\vec{x}_{i_N}^{(n)}\big)B\big(\vec{x}_{i_N}^{(n)} + \lambda^{(n)}\hat{e}\big)\big) \\
&= \frac{1}{n^N}\sum_{\substack{i_1,\ldots,i_N \\ \forall_{k\neq l} i_k\neq i_l}} \varphi\big(A\big(\vec{x}_{i_1}^{(n)}\big)B\big(\vec{x}_{i_1}^{(n)} + \lambda^{(n)}\hat{e}\big)\cdots A\big(\vec{x}_{i_N}^{(n)}\big)B\big(\vec{x}_{i_N}^{(n)} + \lambda^{(n)}\hat{e}\big)\big) \\
&\quad + \frac{1}{n^N}\sum_{\substack{i_1,\ldots,i_N \\ \exists_{k\neq l} s.t. i_k = i_l}} \varphi\big(A\big(\vec{x}_{i_1}^{(n)}\big)B\big(\vec{x}_{i_1}^{(n)} + \lambda^{(n)}\hat{e}\big)\ldots A\big(\vec{x}_{i_N}^{(n)}\big)B\big(\vec{x}_{i_N}^{(n)} + \lambda^{(n)}\hat{e}\big)\big) \\
&\leqslant \frac{1}{n^N}\sum_{\substack{i_1,\ldots,i_N \\ \forall_{k\neq l} i_k\neq i_l}} \big\|P_E A\big(\vec{x}_{i_1}^{(n)}\big)B\big(\vec{x}_{i_1}^{(n)} + \lambda^{(n)}\hat{e}\big)\ldots A\big(\vec{x}_{i_N}^{(n)}\big)B\big(\vec{x}_{i_N}^{(n)} + \lambda^{(n)}\hat{e}\big)P_E\big\| \\
&\quad + \frac{1}{n^N}\sum_{\substack{i_1,\ldots,i_N \\ \exists_{k\neq l} s.t. i_k = i_l}} (\|A\|\|B\|)^N.
\end{aligned} \tag{3.10}$$

By the same reasoning as in case (a) we obtain the estimate

$$\begin{aligned}
|\varphi(\hat{Q}_n)|^N \leqslant{}& \sup_{\substack{A\in\mathfrak{A}_{\mathrm{c}}(\mathcal{O})_1 \\ (\vec{x}_1,\ldots,\vec{x}_{2N})\in\Gamma_{2N,\delta(n)}}} \|P_E A(\vec{x}_1)B(\vec{x}_2)\cdots A(\vec{x}_{2N-1})B(\vec{x}_{2N})P_E\| \\
&+ \frac{c_N}{n}(\|A\|\|B\|)^N.
\end{aligned} \tag{3.11}$$

By taking the limit $n \to \infty$ we conclude the proof of (3.5). □

Now we are ready to prove our main theorem.

Theorem 3.3. *Suppose that condition $C_\flat$ is satisfied. Then there hold the following assertions:*

(a) Let $\omega \in \mathfrak{A}^*$ be a state in the weak closure of $\mathcal{T}_{E,1}$ which is invariant under translations in space. Then $\omega = \omega_0$.*

(b) ω_0 is a pure state.

Proof. In part (a) we proceed similarly as in the proof of theorem 2.2 (b) from [6]: let $\{\varphi_\beta\}_{\beta\in I}$ be a net of functionals from $\mathcal{T}_{E,1}$ approximating ω in the weak* topology and let $A \in \mathfrak{A}_{\mathrm{c}}(\mathcal{O})$

i.e. $\omega_0(A) = 0$. We choose families of points $\{\vec{x}_i\}_1^n$ in $\mathbb{R}^s$ s.t. $\{\vec{x}_i\}_1^n \in \Gamma_{n,\delta(n)}$ for some sequence $\{\delta(n)\}_1^\infty$ which diverges to infinity with $n \to \infty$. We note the following relation

$$\begin{aligned}|\omega(A)| &= \left|\frac{1}{n}\sum_{i=1}^{n}\omega\big(A\big(\vec{x}_i^{(n)}\big)\big)\right| = \lim_{\beta}\left|\frac{1}{n}\sum_{i=1}^{n}\varphi_\beta\big(A\big(\vec{x}_i^{(n)}\big)\big)\right| \\ &\leqslant \sup_{\varphi\in\mathcal{T}_{E,1}}\left|\frac{1}{n}\sum_{i=1}^{n}\varphi\big(A\big(\vec{x}_i^{(n)}\big)\big)\right| \underset{n\to\infty}{\longrightarrow} 0,\end{aligned} \tag{3.12}$$

where in the first step we made use of the fact that the state ω is invariant under translations in space and in the last step we made use of lemma 3.2 (a). Since local algebras are norm dense in the global algebra $\mathfrak{A}$, we conclude that $\ker\omega_0 \subset \ker\omega$ and therefore the two states are equal.

Let us now proceed to part (b) of the theorem. In order to show purity of ω_0, it suffices to verify that for any $A, B \in \mathfrak{A}_c(\mathcal{O})$, some unit vector $\hat{e} \in \mathbb{R}^s$ and some sequence of real numbers $\{\lambda^{(n)}\}_1^\infty$ s.t. $\lambda^{(n)} \underset{n\to\infty}{\longrightarrow} \infty$ there holds

$$\lim_{n\to\infty}\omega_0(AB(\lambda^{(n)}\hat{e})) = 0. \tag{3.13}$$

To this end, we pick a net $\{\varphi_\beta\}_{\beta\in I}$ of functionals from $\mathcal{T}_{E,1}$, approximating ω_0 in the weak* topology. (Such nets exist, since ω_0 is energetically accessible). Next, we choose families of points $\{\vec{x}_i^{(n)}\}_1^n$ as in part (b) of lemma 3.2 and compute

$$\begin{aligned}|\omega_0(AB(\lambda^{(n)}\hat{e}))| &= \left|\frac{1}{n}\sum_{i=1}^{n}\omega_0\big(A\big(\vec{x}_i^{(n)}\big)B\big(\vec{x}_i^{(n)}+\lambda^{(n)}\hat{e}\big)\big)\right| \\ &= \lim_{\beta}\left|\frac{1}{n}\sum_{i=1}^{n}\varphi_\beta\big(A\big(\vec{x}_i^{(n)}\big)B\big(\vec{x}_i^{(n)}+\lambda^{(n)}\hat{e}\big)\big)\right| \\ &\leqslant \sup_{\varphi\in\mathcal{T}_{E,1}}\left|\frac{1}{n}\sum_{i=1}^{n}\varphi\big(A\big(\vec{x}_i^{(n)}\big)B\big(\vec{x}_i^{(n)}+\lambda^{(n)}\hat{e}\big)\big)\right| \underset{n\to\infty}{\longrightarrow} 0,\end{aligned} \tag{3.14}$$

which proves relation (3.13). □

As a corollary we obtain the convergence of states of bounded energy to the vacuum state under large spacelike or timelike translations. (It is an interesting open problem if this corollary holds also for lightlike directions).

Corollary 3.4. *Let condition $C_\flat$ be satisfied. Then, for any state $\omega \in \mathcal{T}_E$, $E \geqslant 0$, and a spacelike or timelike unit vector $\hat{e} \in \mathbb{R}^{s+1}$, there holds*

$$\lim_{\lambda\to\infty}\omega_{\lambda\hat{e}}(A) = \omega_0(A) \text{ for } A \in \mathfrak{A}. \tag{3.15}$$

Proof. First, let $\hat{e}$ be a spacelike vector. Then, by locality, $\{A(\lambda\hat{e})\}_{\lambda>0}$ is a central net in $\mathfrak{A}$ for any $A \in \mathfrak{A}$. Thus, by the irreducibility assumption, its limit points as $\lambda \to \infty$ in the weak* topology of $B(\mathcal{H})$ are multiples of the identity. It follows that the limit points of the net $\{\omega_{\lambda\hat{e}}\}_{\lambda>0}$ are translationally invariant and energetically accessible states. By theorem 3.3 (a), the only such state is ω_0.

If $\hat{e}$ is a timelike vector, the proof relies on an observation due to Buchholz that limit points of $\{\omega_{\lambda\hat{e}}\}_{\lambda>0}$ as $\lambda \to \infty$ are invariant under translations in some spacelike hyperplane as a result of Lorentz covariance. (See lemma 2.3 of [6]). Then it follows from theorem 2.2 (a) of [6] that these limit points are vacuum states. They coincide with ω_0 due to theorem 3.3 (a) above. □

To conclude this survey of applications of condition $C_\flat$, we recall from [6] another physically meaningful procedure for the preparation of vacuum states: it is to construct states with increasingly sharp values of energy and momentum, and exploit the uncertainty principle. Let $P_{(p,r)}$ be the spectral projection corresponding to the ball of radius r centered around point p in the energy–momentum spectrum and $\mathcal{T}_{(p,r)} = P_{(p,r)}\mathcal{T}P_{(p,r)}$. Proceeding analogously as in proposition 2.5 of [6] and exploiting relation (3.4), we obtain the following result.

Proposition 3.5. *Suppose that condition $C_\flat$ is satisfied. Then, for any $p \in \overline{V}_+$ and double cone $\mathcal{O}$, there holds*

$$\lim_{r\to 0} \sup_{\substack{\varphi\in\mathcal{T}_{(p,r),1}\\ A\in\mathfrak{A}(\mathcal{O})_1}} |\varphi(A) - \varphi(I)\omega_0(A)| = 0. \tag{3.16}$$

4. Condition $C_\flat$ (b) in massive scalar free field theory

We showed in section 2 that the qualitative part (a) of condition $C_\flat$ holds in all theories satisfying condition $C_\sharp$, in particular in (massive and massless) scalar free field theory in physical spacetime [10]. Moreover, we argued in section 1 that in physically meaningful, massive theories there should also hold the strengthened, quantitative part (b) of this condition. As we demonstrated in section 3, this quantitative refinement has a number of interesting consequences pertaining to the vacuum structure. It is the goal of the present section to illustrate the mechanism which enforces condition $C_\flat$ (b) by a direct computation in a theory of massive, non-interacting particles.

To establish notation, we recall some basic facts concerning free scalar field theory in s space dimensions: the Hilbert space $\mathcal{H}$ is the Fock space over $L^2(\mathbb{R}^s, d^s p)$. To a double cone $\mathcal{O}$, whose base is the s-dimensional ball $\mathcal{O}_r$ of radius r, centered at the origin, there correspond the closed subspaces $\mathcal{L}^\pm := [\omega^{\mp\frac{1}{2}}\widetilde{D}(\mathcal{O}_r)]$ and we denote the respective projections by the same symbols. Defining J to be the complex conjugation in configuration space, we introduce the real linear subspace

$$\mathcal{L} = (1 + J)\mathcal{L}^+ + (1 - J)\mathcal{L}^- \tag{4.1}$$

and the corresponding von Neumann algebra

$$\mathfrak{A}(\mathcal{O}) = \{W(f) | f \in \mathcal{L}\}'', \tag{4.2}$$

where $W(f) = \mathrm{e}^{\mathrm{i}(a^*(f)+a(f))}$. The vacuum state ω_0 is induced by the Fock space vacuum Ω, i.e. $\omega_0(\cdot) = (\Omega|\cdot\,\Omega)$, and there holds

$$\omega_0(W(f)) = \mathrm{e}^{-\frac{1}{2}\|f\|^2}. \tag{4.3}$$

The unitary representation of translations has the following form in the single-particle space:

$$(U_1(x)f)(\vec{p}) = \mathrm{e}^{\mathrm{i}(\omega(\vec{p})t-\vec{p}\vec{x})} f(\vec{p}) =: f_x(\vec{p}), \tag{4.4}$$

where $x = (t, \vec{x}), \omega(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$ and we assume that $m > 0$. The translation automorphisms are given by $\alpha_x(\cdot) = U(x) \cdot U(x)^*$, where $U(x)$ is the second quantization of $U_1(x)$. With the help of translations we define local algebras attached to double cones centered at any point of spacetime. Our task is to show that the resulting local net satisfies condition $C_\flat$.

Theorem 4.1. *Massive scalar-free field theory satisfies condition $C_\flat$.*

□

Proof. The main ingredient of the proof is the following elementary evaluation of the N-linear form $\Pi_{E,N,\delta}(\varphi,\underline{\vec{x}})$, where $\varphi\in\mathcal{T}_{E,1}$, $\underline{\vec{x}}\in\Gamma_{N,\delta}$, on the generating elements of $\mathfrak{A}_{\rm c}(\mathcal{O})$

$$\begin{aligned}
&\Pi_{E,N,\delta}(\varphi,\underline{\vec{x}})(\{W(f_1)-\omega_0(W(f_1))I\}\times\cdots\times\{W(f_N)-\omega_0(W(f_N))I\})\\
&\quad=\varphi\big((W\big(f_{1,\vec{x}_1}\big)-\omega_0(W(f_1))I)\cdots\big(W\big(f_{N,\vec{x}_N}\big)-\omega_0(W(f_N))I\big)\big)\\
&\quad=\sum_{R_1,R_2}(-1)^{|R_2|}\mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{|R_2|}\|f_{j_k}\|^2}\varphi\big(W\big(f_{i_1,\vec{x}_{i_1}}+\cdots+f_{i_{|R_1|},\vec{x}_{i_{|R_1|}}}\big)\big)\\
&\quad=\sum_{R_1,R_2}(-1)^{|R_2|}\mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{N}\|f_k\|^2}\mathrm{e}^{-\sum_{1\leqslant k<l\leqslant|R_1|}\mathrm{Re}\langle f_{i_k,\vec{x}_{i_k}}|f_{i_l,\vec{x}_{i_l}}\rangle}\\
&\qquad\cdot\varphi\big(:W\big(f_{i_1,\vec{x}_{i_1}}\big)\cdots W\big(f_{i_{|R_1|},\vec{x}_{i_{|R_1|}}}\big):\big)\\
&\quad=\sum_{R_1,R_2}(-1)^{|R_2|}\mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{N}\|f_k\|^2}\big(\mathrm{e}^{-\sum_{1\leqslant k<l\leqslant|R_1|}\mathrm{Re}\langle f_{i_k,\vec{x}_{i_k}}|f_{i_l,\vec{x}_{i_l}}\rangle}-1\big)\\
&\qquad\cdot\varphi\big(:W\big(f_{i_1,\vec{x}_{i_1}}\big)\cdots W\big(f_{i_{|R_1|},\vec{x}_{i_{|R_1|}}}\big):\big)\\
&\qquad+\mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{N}\|f_k\|^2}\sum_{R_1,R_2}(-1)^{|R_2|}\varphi\big(:W\big(f_{i_1,\vec{x}_{i_1}}+\cdots+f_{i_{|R_1|},\vec{x}_{i_{|R_1|}}}\big):\big),
\end{aligned}\tag{4.5}$$

where the sum extends over all partitions $R_1=(i_1,\ldots,i_{|R_1|})$, $R_2=(j_1,\ldots,j_{|R_2|})$ of an N-element set into two, possibly improper, ordered subsets. (If the condition $1\leqslant k<l\leqslant|R_1|$ is empty, the corresponding sum is understood to be zero). In the second step, we made use of the fact that the Weyl operators are localized in spacelike separated regions. In the third step, we applied the identity $W(f)=\mathrm{e}^{-\frac{1}{2}\|f\|^2}:W(f):$ and in the last step, we divided the expression into two parts: the first part tends to zero for large spacelike separations, due to the decay of $\langle f_{1,x_1}|f_{2,x_2}\rangle$ when x_1-x_2 tends to spacelike infinity. In the next lemma, we show that the last sum on the rhs of (4.5) vanishes for $N>2\frac{E}{m}$, so we can omit this last term in the subsequent discussion.

Lemma 4.2. *Let $E\geqslant0$, $\varphi\in\mathcal{T}_E$ and $N>2\frac{E}{m}$ be a natural number. Then there holds*

$$S:=\sum_{R_1,R_2}(-1)^{|R_2|}\varphi(:W(f_{i_1,\vec{x}_{i_1}}+\cdots+f_{i_{|R_1|},\vec{x}_{i_{|R_1|}}}):)=0,\tag{4.6}$$

where the sum extends over all partitions of an N-element set into disjoint sets R_1, R_2.

Proof. For any $f\in L^2(\mathbb{R}^s,d^sp)$ we introduce the map $M(f):B(\mathcal{H})\to B(\mathcal{H})$ given by

$$M(f)(C)=P_E\mathrm{e}^{\mathrm{i}a^*(f)}C\mathrm{e}^{\mathrm{i}a(f)}P_E,\qquad C\in B(\mathcal{H}).\tag{4.7}$$

The exponentials are defined by their Taylor expansions which are finite (in the massive theory) due to the energy projections. The range of $M(f)$ belongs to $B(\mathcal{H})$ due to the energy bounds [10] which in the massive case give

$$\|a(f_1)\cdots a(f_n)P_E\|\leqslant\left(\frac{E}{m}\right)^{\frac{n}{2}}\|f_1\|\cdots\|f_n\|\tag{4.8}$$

for any $f_1,\ldots,f_n\in L^2(\mathbb{R}^s,d^sp)$. Making use of the fact that $\mathrm{e}^{\mathrm{i}a(f)}P_E=P_E\mathrm{e}^{\mathrm{i}a(f)}P_E$, we obtain $M(f_1)M(f_2)=M(f_2)M(f_1)$ for any $f_1,f_2\in L^2(\mathbb{R}^s,d^sp)$. Moreover, $M(0)(C)=P_ECP_E$ for any $C\in B(\mathcal{H})$. We denote by $\hat{I}$ the identity operator acting from $B(\mathcal{H})$ to $B(\mathcal{H})$. There clearly holds

$$\begin{aligned}
S&=\sum_{R_1,R_2}(-1)^{|R_2|}\varphi\big(M\big(f_{i_1,\vec{x}_{i_1}}\big)\cdots M\big(f_{i_{|R_1|},\vec{x}_{i_{|R_1|}}}\big)(I)\big)\\
&=\varphi\big(\big(M\big(f_{1,\vec{x}_1}\big)-\hat{I}\big)\cdots(M\big(f_{N,\vec{x}_N}\big)-\hat{I})(I)\big)\\
&=\varphi\big(\big(M\big(f_{1,\vec{x}_1}\big)-M(0)\big)\cdots\big(M\big(f_{N,\vec{x}_N}\big)-M(0)\big)(I)\big),
\end{aligned}\tag{4.9}$$

where the last equality holds due to the fact that $\varphi \in \mathcal{T}_E$. Finally, we note that for any $C \in B(\mathcal{H})$

$$\big(M(f) - M(0)\big)(C) = \sum_{k+l\geqslant 1} P_E \frac{(ia^*(f))^k}{k!} C \frac{(ia(f))^l}{l!} P_E. \tag{4.10}$$

Substituting this relation into (4.9) the assertion follows. □

We will exploit relation (4.5) to show that for $N > 2\frac{E}{m}$ the norms of the maps $\Pi_{E,N,\delta}$ tend to zero with $\delta \to \infty$, what entails condition $C_\flat$ in view of lemma 3.1. To this end, we introduce the *-algebra $\mathfrak{A}_0(\mathcal{O})$, generated by finite linear combinations of Weyl operators and denote by $(\mathfrak{A}_0(\mathcal{O})^{\times M})^*$ the space of (not necessarily bounded) M-linear forms on $\mathfrak{A}_0(\mathcal{O})$. We define the maps $\Pi'_{E,M,\delta} : \mathcal{T}_E \times \Gamma_{M,\delta} \to (\mathfrak{A}_0(\mathcal{O})^{\times M})^*$, linear in the first argument, extending by linearity the following expression:

$$\begin{aligned}\Pi'_{E,M,\delta}(\varphi, \underline{\vec{x}})&(W(f_1) \times \cdots \times W(f_M)) \\ &= \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^M \|f_k\|^2} (\mathrm{e}^{-\sum_{1\leqslant i<j\leqslant M} \mathrm{Re}\langle f_{i,\vec{x}_i}|f_{j,\vec{x}_j}\rangle} - 1)\varphi(:W(f_{1,\vec{x}_1} + \cdots + f_{M,\vec{x}_M}):).\end{aligned} \tag{4.11}$$

We obtain from (4.5) the equality valid for $N > 2\frac{E}{m}$

$$\begin{aligned}&\Pi_{E,N,\delta}(\varphi, \underline{\vec{x}})\big(\{W(f_1) - \omega_0(W(f_1))I\} \times \cdots \times \{W(f_N) - \omega_0(W(f_N))I\}\big) \\ &\quad = \sum_{R_1,R_2} (-1)^{|R_2|} \omega_0(W(f_{j_1})) \cdots \omega_0(W(f_{j_{|R_2|}})) \cdot \\ &\qquad \cdot \Pi'_{E,|R_1|,\delta}(\varphi, \underline{\vec{x}})(W(f_{i_1}) \times \cdots \times W(f_{i_{|R_1|}})).\end{aligned} \tag{4.12}$$

To conclude the argument we need the following technical lemma.

Lemma 4.3. For any $M \in \mathbb{N}$, $E \geqslant 0$, double cone $\mathcal{O}$ and sufficiently large $\delta > 0$ (depending on M, E and $\mathcal{O}$) there exist the maps $\Pi''_{E,M,\delta} \in \mathcal{L}(\mathcal{T}_E \times \Gamma_{M,\delta}, (\mathfrak{A}(\mathcal{O})^{\otimes M})_*)$ which have the properties

(a) $\lim_{\delta\to\infty} \|\Pi''_{E,M,\delta}\| = 0$;
(b) $\Pi''_{E,M,\delta}(\varphi, \underline{\vec{x}})(A_1 \otimes \cdots \otimes A_M) = \Pi'_{E,M,\delta}(\varphi, \underline{\vec{x}})(A_1 \times \cdots \times A_M)$ for $A_1, \ldots, A_M \in \mathfrak{A}_0(\mathcal{O})$ and any $(\varphi, \underline{\vec{x}}) \in \mathcal{T}_E \times \Gamma_{M,\delta}$.

In view of this lemma, whose proof is postponed to appendix A, equality (4.12) can now be rewritten as follows, for sufficiently large N, δ and any $A_1, \ldots, A_N \in \mathfrak{A}_c(\mathcal{O})$

$$\Pi_{E,N,\delta}(\varphi, \underline{\vec{x}})(A_1 \times \cdots \times A_N) = \Pi''_{E,N,\delta}(\varphi, \underline{\vec{x}})(A_1 \otimes \cdots \otimes A_N), \tag{4.13}$$

where we made use of the facts that $\omega_0(A_1) = \cdots = \omega_0(A_N) = 0$, $\mathfrak{A}_0(\mathcal{O})$ is dense in $\mathfrak{A}(\mathcal{O})$ in the strong operator topology, and $\Pi''_{E,M,\delta}(\varphi, \underline{\vec{x}})$ is a normal functional on $(\mathfrak{A}(\mathcal{O})^{\otimes M})$. Consequently, for $N > 2\frac{E}{m}$, the map $\Pi_{E,N,\delta}$ shares the properties of $\Pi''_{E,N,\delta}$ stated in lemma 4.3. In addition we know from theorem 1.2 that the maps $\Pi_{E,N,\delta}$ are compact for any $\delta > 0$. Making use of lemma 3.1 we conclude that condition $C_\flat$ is satisfied. □

We remark that the assumption $m > 0$ is used only in one (crucial) step in the above proof, namely to eliminate the last term in relation (4.5) and establish equality (4.13). The properties of the maps $\Pi'_{E,M,\delta}$, stated in lemma 4.3, hold in massless free field theory as well. However, a complete verification argument for condition $C_\flat$ in the massless case has not been found yet.

5. Conclusions

In this work, we introduced and verified in a model the phase space condition $C_\flat$ which encodes localization of physical states in space. More precisely, it says that any coincidence arrangement of spacelike separated observables with vanishing vacuum expectation values gives zero response if the number of observables is much larger than the number of localization centers which form the state. From this physically motivated observation we derived detailed information about the vacuum state: it is pure and unique in the energetically connected component of the state space, it can be prepared with the help of states with increasingly sharp energy–momentum values and appears as a limit of physical states under large spacelike or timelike translations.

This last property corroborates the intuitive picture of spreading of wave packets which prevents the detection of particles with the help of observables of fixed spatial extension. In order to determine the particle content and compute collision cross sections, one has to consider coincidence arrangements of particle detectors, whose responses are suitably rescaled with time [3, 13–15]. The methods developed in the present work are also of relevance to the study of these interesting problems.

Acknowledgments

I would like to thank Professor D Buchholz for suggesting to me the present approach to the study of the vacuum structure and for many valuable discussions in the course of this work. Financial support from Deutsche Forschungsgemeinschaft and Graduiertenkolleg 'Mathematische Strukturen in der modernen Quantenphysik' of the University of Göttingen is gratefully acknowledged. This paper was completed at the TU München under the DFG grant SP181/25-1.

Appendix A. Proof of lemma 4.3

The goal of this appendix is to construct the maps $\Pi''_{E,M,\delta} \in \mathcal{L}(\mathcal{T}_E \times \Gamma_{M,\delta}, (\mathfrak{A}(\mathcal{O})^{\otimes M})_*)$ and verify that they have the properties (a) and (b) specified in lemma 4.3. We will define these maps as norm-convergent sums of rank-one mappings, i.e.

$$\Pi''_{E,M,\delta} = \sum_{i=1}^{\infty} \tau_i S_i, \tag{A.1}$$

where $\tau_i \in (\mathfrak{A}(\mathcal{O})^{\otimes M})_*$ and $S_i \in \mathcal{L}(\mathcal{T}_E \times \Gamma_{M,\delta}, \mathbb{C})$.

In order to construct a suitable family of functionals τ_i, we recall some facts from [16, 17]: given any pair of multi-indices μ^+, μ^- and an orthonormal basis $\{e_i\}_1^\infty$ of J-invariant eigenvectors in the single-particle space $L^2(\mathbb{R}^s, d^s p)$, one defines a normal functional $\tau_{\mu^+,\mu^-} \in B(\mathcal{H})_*$ by the following formula:

$$\tau_{\mu^+,\mu^-}(A) = \left(\frac{1}{2}\right)^{|\mu^+|+|\mu^-|} i^{-|\mu^+|-2|\mu^-|} \sum_{\substack{\alpha^+ + \alpha'^+ + \alpha''^+ = \mu^+ \\ \alpha^- + \alpha'^- + \alpha''^- = \mu^-}} (-1)^{|\alpha'^+|+|\alpha''^-|} \frac{\mu^+!}{\alpha^+!\alpha'^+!\alpha''^+!} \frac{\mu^-!}{\alpha^-!\alpha'^-!\alpha''^-!}$$
$$\cdot (\Omega | a(e)^{\alpha^+ + \alpha^-} a^*(e)^{\alpha'^+ + \alpha'^-} A a^*(e)^{\alpha''^+ + \alpha''^-} \Omega), \tag{A.2}$$

where $A \in B(\mathcal{H})$ and $\alpha^\pm, \alpha'^\pm, \alpha''^\pm$ are multi-indices. It is shown in [16] that these functionals take the following values on the Weyl operators:

$$\tau_{\mu^+,\mu^-}(W(f)) = \mathrm{e}^{-\frac{1}{2}\|f\|^2} \langle e|f^+\rangle^{\mu^+} \langle e|f^-\rangle^{\mu^-}, \tag{A.3}$$

where $f = f^+ + \mathrm{i} f^- \in \mathcal{L}$ and f^+, f^- are the real and imaginary parts of f in configuration space. It is also established there that the norms of these functionals satisfy the bound

$$\|\tau_{\mu^+,\mu^-}\| \leqslant 4^{|\mu^+|+|\mu^-|}(\mu^+!\mu^-!)^{\frac{1}{2}}. \tag{A.4}$$

Turning to the definition of suitable functionals on $B(\mathcal{H})^{\otimes M}$, we introduce M-tuples of multi-indices $\underline{\mu}^{\pm} = (\mu_1^{\pm}, \ldots, \mu_M^{\pm})$ and the corresponding $2M$-multi-indices $\underline{\mu} = (\underline{\mu}^+, \underline{\mu}^-)$. We extend the standard rules of the multi-index notation as follows:

$$|\underline{\mu}| = \sum_{i=1}^{M}(|\mu_i^+| + |\mu_i^-|), \tag{A.5}$$

$$\underline{\mu}! = \prod_{i=1}^{M} \mu_i^+!\mu_i^-!, \tag{A.6}$$

$$\langle e|f\rangle^{\underline{\mu}} = \prod_{i=1}^{M} \langle e|f_i^+\rangle^{\mu_i^+}\langle e|f_i^-\rangle^{\mu_i^-}, \tag{A.7}$$

where $f_1, \ldots, f_M \in \mathcal{L}$. Now for any $2M$-multi-index $\underline{\mu}$ we define a normal functional $\tau_{\underline{\mu}}$ on $B(\mathcal{H})^{\otimes M}$ by the expression

$$\tau_{\underline{\mu}} = \tau_{\mu_1^+,\mu_1^-} \otimes \cdots \otimes \tau_{\mu_M^+,\mu_M^-}. \tag{A.8}$$

From relations (A.3), (A.4) and the polar decomposition of a normal functional [12] one immediately obtains

Lemma A.1. *Let $\{e_i\}_1^\infty$ be an orthonormal basis in $L^2(\mathbb{R}^s, d^s p)$ of J-invariant eigenvectors. The functionals $\tau_{\underline{\mu}} \in (B(\mathcal{H})^{\otimes M})_*$ given by (A.8) have the following properties:*

(a) $\tau_{\underline{\mu}}(W(f_1) \otimes \cdots \otimes W(f_M)) = \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{M}\|f_k\|^2}\langle e|f\rangle^{\underline{\mu}}$;

(b) $\|\tau_{\underline{\mu}}\| \leqslant 4^{|\underline{\mu}|}(\underline{\mu}!)^{\frac{1}{2}}$;

where $f_1, \ldots, f_M \in \mathcal{L}$.

In order to construct a basis $\{e_i\}_1^\infty$ in $L^2(\mathbb{R}^s, d^s p)$ of J-invariant eigenvectors, which is suitable for our purposes, we recall, with certain modifications, some material from the literature. Let Q_E be the projection on states of energy lower than E in the single-particle space. We define the operators $T_E^{\pm} = Q_E\mathcal{L}^{\pm}$ and $T_\kappa^{\pm} = \mathrm{e}^{-\frac{|\omega|^\kappa}{2}}\mathcal{L}^{\pm}$, where $0 < \kappa < 1$. By a slight modification of lemma 3.5 from [10] one finds that these operators satisfy $\|T_E^{\pm}\|_1 < \infty, \|T_\kappa^{\pm}\|_1 < \infty$, where $\|\cdot\|_1$ denotes the trace norm. We define the operator T as follows:

$$T^2 = |T_E^+|^2 + |T_E^-|^2 + |T_\kappa^+|^2 + |T_\kappa^-|^2. \tag{A.9}$$

Making use of the estimate $\|(A+B)^p\|_1 \leqslant \|A^p\|_1 + \|B^p\|_1$, valid for any $0 < p \leqslant 1$ and any pair of positive operators A, B s.t. A^p, B^p are trace class [18], we obtain

$$\|T\|_1 \leqslant \|T_E^+\|_1 + \|T_E^-\|_1 + \|T_\kappa^+\|_1 + \|T_\kappa^-\|_1 < \infty. \tag{A.10}$$

Since T commutes with J, it has a J-invariant orthonormal basis of eigenvectors $\{e_i\}_1^\infty$ and we denote the corresponding eigenvalues by $\{t_i\}_1^\infty$.

Now we proceed to the construction of the functionals $S_i \in \mathcal{L}(\mathcal{T}_E \times \Gamma_{M,\delta}, \mathbb{C})$, to appear in expansion (A.1). Let $\hat{\alpha}^{\pm} = (\alpha_{1,2}^{\pm}, \ldots, \alpha_{M-1,M}^{\pm})$ be $\binom{M}{2}$-tuples of multi-indices and let

$\hat{\alpha} = (\hat{\alpha}^+, \hat{\alpha}^-)$ be the corresponding $2\binom{M}{2}$-multi-index. First, we define the contribution to the functional which is responsible for the correlations between measurements:

$$F_{\hat{\alpha},\hat{\beta}}(\underline{\vec{x}}) = \prod_{1\leqslant i<j\leqslant M} \frac{(-1)^{|\alpha_{i,j}^-|+|\alpha_{i,j}^+|}}{\sqrt{\alpha_{i,j}^+!\beta_{i,j}^+!\alpha_{i,j}^-!\beta_{i,j}^-!}} (\Omega|a(\mathcal{L}^+e_{\vec{x}_i})^{\alpha_{i,j}^+}a^*(\mathcal{L}^+e_{\vec{x}_j})^{\beta_{i,j}^+}\Omega)$$
$$\cdot (\Omega|a(\mathcal{L}^-e_{\vec{x}_i})^{\alpha_{i,j}^-}a^*(\mathcal{L}^-e_{\vec{x}_j})^{\beta_{i,j}^-}\Omega), \tag{A.11}$$

where we use the short-hand notation $\mathcal{L}^\pm e_{i,\vec{x}_j} = U(\vec{x}_j)\mathcal{L}^\pm e_i$. The functionals in question are given by

$$S_{\underline{\mu},\underline{\nu},\hat{\alpha},\hat{\beta}}(\varphi, \underline{\vec{x}}) = \frac{\mathrm{i}^{|\underline{\mu}^+|+|\underline{\nu}^+|+2|\underline{\mu}^-|}}{\underline{\mu}!\underline{\nu}!\sqrt{\hat{\alpha}!\hat{\beta}!}} F_{\hat{\alpha},\hat{\beta}}(\underline{\vec{x}})\varphi(a^*(\mathcal{L}e_{\vec{x}})^{\underline{\mu}}a(\mathcal{L}e_{\vec{x}})^{\underline{\nu}}), \tag{A.12}$$

where $\varphi \in \mathcal{T}_E$ and $\underline{\vec{x}} \in \Gamma_{M,\delta}$. The norms of these functionals satisfy the bound, stated in the following lemma, whose proof is postponed to appendix B.

Lemma A.2. *The functionals $S_{\underline{\mu},\underline{\nu},\hat{\alpha},\hat{\beta}} \in \mathcal{L}(\mathcal{T}_E \times \Gamma_{M,\delta}, \mathbb{C})$, given by (A.12), satisfy the following estimates:*

$$\|S_{\underline{\mu},\underline{\nu},\hat{\alpha},\hat{\beta}}\| \leqslant \left(\frac{M_E^{\frac{1}{2}(|\underline{\mu}|+|\underline{\nu}|)}}{\underline{\mu}!\underline{\nu}!} t^{\underline{\mu}+\underline{\nu}}\right)\left(\frac{1}{\sqrt{\hat{\alpha}!\hat{\beta}!}}\sqrt{\frac{|\hat{\alpha}^+|!|\hat{\alpha}^-|!|\hat{\beta}^+|!|\hat{\beta}^-|!}{\hat{\alpha}!\hat{\beta}!}} g(\delta)^{|\hat{\alpha}|+|\hat{\beta}|} t^{\hat{\alpha}+\hat{\beta}}\right), \tag{A.13}$$

where $M_E = \frac{E}{m}$, $\{t_i\}_1^\infty$ are the eigenvalues of the operator T given by (A.9) and the function g, which is independent of $\hat{\alpha}$ and $\hat{\beta}$, satisfies $\lim_{\delta\to\infty} g(\delta) = 0$.

Given the estimates from lemmas A.1 (b) and A.2 we can proceed to the study of convergence properties of expansion (A.1). For this purpose we need some notation: for any pair of $\binom{M}{2}$-tuples of multi-indices $\hat{\alpha}^\pm = (\alpha_{1,2}^\pm, \ldots, \alpha_{M-1,M}^\pm)$ we define the associated M-tuples of multi-indices $\underset{\rightarrow}{\hat{\alpha}}^\pm$, $\underset{\leftarrow}{\hat{\alpha}}^\pm$ as follows:

$$\underset{\rightarrow}{\hat{\alpha}}_i^\pm = \sum_{\substack{1<j\leqslant M\\ i<j}} \alpha_{i,j}^\pm, \tag{A.14}$$

$$\underset{\leftarrow}{\hat{\alpha}}_i^\pm = \sum_{\substack{1\leqslant j<M,\\ j<i}} \alpha_{j,i}^\pm, \tag{A.15}$$

where $i \in \{1, \ldots, M\}$. The corresponding $2M$-multi-indices are denoted by $\underset{\rightarrow}{\hat{\alpha}} = (\underset{\rightarrow}{\hat{\alpha}}^+, \underset{\rightarrow}{\hat{\alpha}}^-)$, $\underset{\leftarrow}{\hat{\alpha}} = (\underset{\leftarrow}{\hat{\alpha}}^+, \underset{\leftarrow}{\hat{\alpha}}^-)$. The relevant estimate is stated in the following lemma, whose proof is given in appendix B.

Lemma A.3. *The functionals $\tau_{\underline{\mu}} \in (\mathfrak{A}(\mathcal{O})^{\otimes N})_*$ and $S_{\underline{\mu},\underline{\nu},\hat{\alpha},\hat{\beta}} \in \mathcal{L}(\mathcal{T}_E \times \Gamma_{M,\delta}, \mathbb{C})$ satisfy*

$$\sum_{\substack{\underline{\mu},\underline{\nu}\\ \hat{\alpha},\hat{\beta}\\ (|\hat{\alpha}|,|\hat{\beta}|)\neq(0,0)}} \|\tau_{\underline{\mu}+\underline{\nu}+\underset{\rightarrow}{\hat{\alpha}}+\underset{\leftarrow}{\hat{\beta}}}\|\|S_{\underline{\mu},\underline{\nu},\hat{\alpha},\hat{\beta}}\| < \infty \tag{A.16}$$

for sufficiently large $\delta > 0$, depending on M, E and the double cone $\mathcal{O}$. Moreover, the above sum tends to zero with $\delta \to \infty$.

After this preparation we proceed to the main part of this appendix.

Proof of lemma 4.3. We define $\Pi''_{E,M,\delta} \in \mathcal{L}(\mathcal{T}_E \times \Gamma_{M,\delta}, (\mathfrak{A}(\mathcal{O})^{\otimes M})_*)$ as follows:

$$\Pi''_{E,M,\delta}(\varphi,\underline{\vec{x}}) = \sum_{\substack{\underline{\mu},\underline{\nu}\\ \hat{\alpha},\hat{\beta}\\ (|\hat{\alpha}|,|\hat{\beta}|)\neq(0,0)}} \tau_{\underline{\mu}+\underline{\nu}+\underset{\rightarrow}{\hat{\alpha}}+\underset{\leftarrow}{\hat{\beta}}} S_{\underline{\mu},\underline{\nu},\hat{\alpha},\hat{\beta}}(\varphi,\underline{\vec{x}}). \tag{A.17}$$

In view of lemma A.3 this map is well defined for sufficiently large $\delta > 0$ and satisfies $\lim_{\delta\to\infty} \|\Pi''_{E,M,\delta}\| = 0$ as required in part (a) of lemma 4.3. In order to verify part (b), it suffices to show that for any $f_1,\ldots,f_M \in \mathcal{L}$

$$\begin{aligned}\Pi''_{E,M,\delta}(\varphi,\underline{\vec{x}})(W(f_1)\otimes\cdots\otimes W(f_M)) &= \Pi'_{E,M,\delta}(\varphi,\underline{\vec{x}})(W(f_1)\times\cdots\times W(f_M))\\ &= \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{M}\|f_k\|^2}\left(\mathrm{e}^{-\sum_{1\leqslant i<j\leqslant M}(\langle f^+_{i,\vec{x}_i}|f^+_{j,\vec{x}_j}\rangle+\langle f^-_{i,\vec{x}_i}|f^-_{j,\vec{x}_j}\rangle)}-1\right)\\ &\quad\cdot\varphi(:W(f_{1,\vec{x}_1}+\cdots+f_{M,\vec{x}_M}):),\end{aligned} \tag{A.18}$$

where the second equality restates the definition of the map $\Pi'_{E,M,\delta}$ given by formula (4.11). The lhs can be evaluated making use of lemma A.1 (a) and definition (A.12)

$$\begin{aligned}&\Pi''_{E,N,\delta}(\varphi,\underline{\vec{x}})(W(f_1)\times\cdots\times W(f_N))\\ &\quad= \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{M}\|f_k\|^2}\sum_{\substack{\underline{\mu},\underline{\nu}\\ \hat{\alpha},\hat{\beta}\\ (|\hat{\alpha}|,|\hat{\beta}|)\neq(0,0)}}\frac{i^{|\underline{\mu}^+|+|\underline{\nu}^+|+2|\underline{\mu}^-|}}{\underline{\mu}!\underline{\nu}!\sqrt{\hat{\alpha}!\hat{\beta}!}}\langle e|f\rangle^{\underline{\mu}+\underline{\nu}+\underset{\rightarrow}{\hat{\alpha}}+\underset{\leftarrow}{\hat{\beta}}}F_{\hat{\alpha},\hat{\beta}}(\underline{\vec{x}})\varphi(a^*(\mathcal{L}e_{\vec{x}})^{\underline{\mu}}a(\mathcal{L}e_{\vec{x}})^{\underline{\nu}}).\end{aligned} \tag{A.19}$$

First, we consider the sum w.r.t. $\underline{\mu},\underline{\nu}$. There holds

$$\sum_{\underline{\mu},\underline{\nu}}\frac{i^{|\underline{\mu}^+|+|\underline{\nu}^+|+2|\underline{\mu}^-|}}{\underline{\mu}!\underline{\nu}!}\langle e|f\rangle^{\underline{\mu}+\underline{\nu}}\varphi(a^*(\mathcal{L}e_{\vec{x}})^{\underline{\mu}}a(\mathcal{L}e_{\vec{x}})^{\underline{\nu}}) = \varphi(:W(f_{1,\vec{x}_1}+\cdots+f_{M,\vec{x}_M}):), \tag{A.20}$$

as one can verify by expanding the normal-ordered Weyl operator on the rhs into the power series of creation and annihilation operators of the functions $f^{\pm}_{j,\vec{x}_j}$, expanding each such function in the orthonormal basis $\{e_i\}_1^\infty$ and making use of the multinomial formula

$$a^{(*)}(f^{\pm}_{j,\vec{x}_j})^{m_j^{\pm}} = \sum_{\mu_j^{\pm},|\mu_j^{\pm}|=m_j^{\pm}}\frac{m_j^{\pm}!}{\mu_j^{\pm}!}\langle e|f_j\rangle^{\mu_j^{\pm}}a^{(*)}(\mathcal{L}^+e_{\vec{x}_j})^{\mu_j^{\pm}}. \tag{A.21}$$

The sum w.r.t. $\hat{\alpha},\hat{\beta}$ in (A.19) gives

$$\sum_{\substack{\hat{\alpha},\hat{\beta}\\ (|\hat{\alpha}|,|\hat{\beta}|)\neq(0,0)}}\frac{1}{\sqrt{\hat{\alpha}!\hat{\beta}!}}\langle e|f\rangle^{\underset{\rightarrow}{\hat{\alpha}}+\underset{\leftarrow}{\hat{\beta}}}F_{\hat{\alpha},\hat{\beta}}(\underline{\vec{x}}) = \left(\prod_{1\leqslant i<j\leqslant M}\mathrm{e}^{-\langle f^+_{i,\vec{x}_i}|f^+_{j,\vec{x}_j}\rangle}\mathrm{e}^{-\langle f^-_{i,\vec{x}_i}|f^-_{j,\vec{x}_j}\rangle}\right)-1. \tag{A.22}$$

This relation can be verified by expanding the exponential functions on the rhs into Taylor series, making use of the identity

$$\langle f^{\pm}_{i,\vec{x}_i}|f^{\pm}_{j,\vec{x}_j}\rangle^{k^{\pm}_{i,j}} = \frac{\left(\Omega|a\left(f^{\pm}_{i,\vec{x}_i}\right)^{k^{\pm}_{i,j}}a^*\left(f^{\pm}_{j,\vec{x}_j}\right)^{k^{\pm}_{i,j}}\Omega\right)}{k^{\pm}_{i,j}!} \tag{A.23}$$

and applying to the resulting expression expansions (A.21). Comparing (A.22) and (A.20) with (A.18) we conclude the proof of lemma 4.3. □

Appendix B. Some technical proofs

In this appendix, we provide proofs of lemmas A.2 and A.3 which we used in appendix A to prove lemma 4.3.

The key ingredient of our proof of lemma A.2 is the observation that when the spatial distance between two local operators is large, then the energy transfer between them is heavily damped. We exploited this idea in section 2, where it was encoded in lemma 2.1. In the present context it is more convenient to use the following result, which is a variant of lemma 2.3 of [19].

Lemma B.1. *Let $\delta > 0$. Then there exists some continuous function $h(\omega)$, $\omega \in \mathbb{R}$ which decreases almost exponentially, i.e.*

$$\sup_{\omega} |h(\omega)|\mathrm{e}^{|\omega|^{\kappa}} < \infty \text{ for } 0 < \kappa < 1, \tag{B.1}$$

and which has the following property: for any pair of operators A, B which have, together with their adjoints, a common, invariant, stable under time translations dense domain containing Ω and satisfy $[A(t), B] = 0$ for $|t| < \delta$, there holds the identity

$$(\Omega|AB\Omega) = \tfrac{1}{2}\{(\Omega|Ah(\delta H)B\Omega) + (\Omega|Bh(\delta H)A\Omega)\}. \tag{B.2}$$

With the help of this result we prove the following key lemma which will help us to control the correlation terms $F_{\hat{\alpha},\hat{\beta}}$.

Lemma B.2. *Let $\delta > 0$, $(\vec{x}, \vec{y}) \in \Gamma_{2,\delta}$, $\{e_i\}_1^{\infty}$ be the basis of the J-invariant eigenvectors of the operator T given by (A.9), let $\{t_i\}_1^{\infty}$ be the corresponding eigenvalues and let α, β be multi-indices. Then there holds, for any combination of $\pm$ signs,*

$$|(\Omega|a(\mathcal{L}^{\pm}e_{\vec{x}})^{\alpha}a^*(\mathcal{L}^{\pm}e_{\vec{y}})^{\beta}\Omega)| \leqslant \sqrt{|\alpha|!|\beta|!}\,g(\delta)^{|\alpha|+|\beta|}t^{\alpha+\beta}, \tag{B.3}$$

where the function g is independent of α, β and satisfies $\lim_{\delta\to\infty} g(\delta) = 0$.

Proof. We consider here only the (++) case, as the remaining cases are treated analogously. We define the operators $\phi_+(e_i) = a^*(\mathcal{L}^+e_i) + a(\mathcal{L}^+e_i)$ and their translates $\phi_+(e_i)(\vec{x}) = U(\vec{x})\phi_+(e_i)U(\vec{x})^{-1}$. Since the projection $\mathcal{L}^+$ commutes with J and $Je_i = e_i$, these operators are just the (canonical) fields of massive scalar free field theory. Since $\delta > 0$, locality guarantees that $\phi_+(e_i)(\vec{x})$ and $\phi_+(e_j)(\vec{y})$ satisfy the assumptions of lemma B.1. Therefore, we obtain

$$\begin{aligned}\langle\mathcal{L}^+e_{i,\vec{x}}|\mathcal{L}^+e_{j,\vec{y}}\rangle &= (\Omega|\phi_+(e_i)(\vec{x})\phi_+(e_j)(\vec{y})\Omega)\\ &= \tfrac{1}{2}((\Omega|\phi_+(e_i)(\vec{x})h(\delta H)\phi_+(e_j)(\vec{y})\Omega) + (\Omega|\phi_+(e_j)(\vec{y})h(\delta H)\phi_+(e_i)(\vec{x})\Omega))\\ &= \tfrac{1}{2}(\langle\mathcal{L}^+e_{i,\vec{x}}|h(\delta\omega)\mathcal{L}^+e_{j,\vec{y}}\rangle + \langle\mathcal{L}^+e_{j,\vec{y}}|h(\delta\omega)\mathcal{L}^+e_{i,\vec{x}}\rangle).\end{aligned} \tag{B.4}$$

Making use of this result, exploiting the fact that the lhs of (B.3) vanishes for $|\alpha| \neq |\beta|$ and setting $|\alpha| = |\beta| = k$, we get

$$\begin{aligned}(\Omega|a(\mathcal{L}^+e_{\vec{x}})^{\alpha}a^*(\mathcal{L}^+e_{\vec{y}})^{\beta}\Omega) &= (\Omega|a(\mathcal{L}^+e_{i_1,\vec{x}})\cdots a(\mathcal{L}^+e_{i_k,\vec{x}})a^*(\mathcal{L}^+e_{j_1,\vec{y}})\cdots a^*(\mathcal{L}^+e_{j_k,\vec{y}})\Omega)\\ &= \sum_{\sigma\in S_k}\langle\mathcal{L}^+e_{i_1,\vec{x}}|\mathcal{L}^+e_{j_{\sigma_1},\vec{y}}\rangle\cdots\langle\mathcal{L}^+e_{i_k,\vec{x}}|\mathcal{L}^+e_{j_{\sigma_k},\vec{y}}\rangle\\ &= \sum_{\sigma\in S_k}\frac{1}{2}\big(\langle\mathcal{L}^+e_{i_1,\vec{x}}|h(\delta\omega)\mathcal{L}^+e_{j_{\sigma_1},\vec{y}}\rangle + \langle\mathcal{L}^+e_{j_{\sigma_1},\vec{y}}|h(\delta\omega)\mathcal{L}^+e_{i_1,\vec{x}}\rangle\big)\\ &\quad\cdots\frac{1}{2}\big(\langle\mathcal{L}^+e_{i_k,\vec{x}}|h(\delta\omega)\mathcal{L}^+e_{j_{\sigma_k},\vec{y}}\rangle + \langle\mathcal{L}^+e_{j_{\sigma_k},\vec{y}}|h(\delta\omega)\mathcal{L}^+e_{i_k,\vec{x}}\rangle\big),\end{aligned} \tag{B.5}$$

where the sum extends over all permutations of a k-element set. For any $0 < \kappa < 1$ there holds $c_h^2 := \sup_\omega |h(\omega)\mathrm{e}^{|\omega|^\kappa}| < \infty$. Consequently, we get

$$\begin{aligned}|\langle \mathcal{L}^+ e_{i,\bar{x}}|h(\delta\omega)\mathcal{L}^+ e_{j,\bar{y}}\rangle| &= |\langle \mathcal{L}^+ e_{i,\bar{x}}|h(\delta\omega)\mathrm{e}^{(\delta|\omega|)^\kappa}\mathrm{e}^{-(\delta^\kappa-1)|\omega|^\kappa}\mathrm{e}^{-|\omega|^\kappa}\mathcal{L}^+ e_{j,\bar{y}}\rangle| \\ &\leqslant c_h^2 \mathrm{e}^{-(\delta^\kappa-1)m^\kappa}\|\mathrm{e}^{-\frac{|\omega|^\kappa}{2}}\mathcal{L}^+ e_i\|\|\mathrm{e}^{-\frac{|\omega|^\kappa}{2}}\mathcal{L}^+ e_j\|. \end{aligned} \tag{B.6}$$

Finally, we note that $\|\mathrm{e}^{-\frac{|\omega|^\kappa}{2}}\mathcal{L}^+ e_i\| = \|T_\kappa^+ e_i\| \leqslant \|T e_i\| = t_i$ and the claim follows. □

After this preparation we proceed to the proof of lemma A.2.

Proof of lemma A.2. Exploiting the energy bounds [10] (see estimate (4.8) above), we obtain

$$\begin{aligned}|\varphi(a^*(\mathcal{L}e_{\bar{x}})^{\underline{\mu}}a(\mathcal{L}e_{\bar{x}})^{\underline{\nu}})| &\leqslant M_E^{\frac{1}{2}(|\underline{\mu}|+|\underline{\nu}|)}\|Q_E\mathcal{L}e\|^{\underline{\mu}}\|Q_E\mathcal{L}e\|^{\underline{\nu}} \\ &\leqslant M_E^{\frac{1}{2}(|\underline{\mu}|+|\underline{\nu}|)}t^{\underline{\mu}+\underline{\nu}}.\end{aligned} \tag{B.7}$$

Next, with the help of lemma B.2 we analyze the expressions $F_{\hat{\alpha},\hat{\beta}}$ given by (A.11)

$$\begin{aligned}|F_{\hat{\alpha},\hat{\beta}}(\vec{\underline{x}})| &\leqslant \prod_{1\leqslant i<j\leqslant M}\sqrt{\frac{|\alpha_{i,j}^+|!|\beta_{i,j}^+|!|\alpha_{i,j}^-|!|\beta_{i,j}^-|!}{\alpha_{i,j}^+!\beta_{i,j}^+!\alpha_{i,j}^-!\beta_{i,j}^-!}}(g(\delta)t)^{\alpha_{i,j}^++\beta_{i,j}^++\alpha_{i,j}^-+\beta_{i,j}^-} \\ &\leqslant \sqrt{\frac{|\hat{\alpha}^+|!|\hat{\alpha}^-|!|\hat{\beta}^+|!|\hat{\beta}^-|!}{\hat{\alpha}!\hat{\beta}!}}g(\delta)^{|\hat{\alpha}|+|\hat{\beta}|}t^{\hat{\alpha}+\hat{\beta}},\end{aligned} \tag{B.8}$$

where we made use of the estimate $\prod_{1\leqslant i<j\leqslant M}|\alpha_{i,j}^+|! \leqslant (\sum_{1\leqslant i<j\leqslant M}|\alpha_{i,j}^+|)! = |\hat{\alpha}^+|!$. Altogether, combining (B.7) and (B.8), we obtain from (A.12) the bound (A.13). □

We conclude this appendix with a proof of lemma A.3.

Proof of lemma A.3. First, we estimate the norms of the functionals $\tau_{\underline{\mu}+\underline{\nu}+\underset{\rightarrow}{\hat{\alpha}}+\underset{\leftarrow}{\hat{\beta}}}$. Making use of the bound stated in lemma A.1 (b) and of the fact that $(a+b+c)! \leqslant 3^{a+b+c}a!b!c!$ for any $a, b, c \in \mathbb{N}_0$, which follows from properties of the multinomial coefficients, we get

$$\begin{aligned}\|\tau_{\underline{\mu}+\underline{\nu}+\underset{\rightarrow}{\hat{\alpha}}+\underset{\leftarrow}{\hat{\beta}}}\| &\leqslant 4^{|\underline{\mu}|+|\underline{\nu}|+|\underset{\rightarrow}{\hat{\alpha}}|+|\underset{\leftarrow}{\hat{\beta}}|}\sqrt{(\underline{\mu}+\underline{\nu}+\underset{\rightarrow}{\hat{\alpha}}+\underset{\leftarrow}{\hat{\beta}})!} \\ &\leqslant \left((4\sqrt{3})^{|\underline{\mu}|+|\underline{\nu}|}\sqrt{(\underline{\mu}+\underline{\nu})!}\right)\left((4\sqrt{3})^{|\hat{\alpha}|+|\hat{\beta}|}\sqrt{\underset{\rightarrow}{\hat{\alpha}}!\underset{\leftarrow}{\hat{\beta}}!}\right),\end{aligned} \tag{B.9}$$

where we noted that $|\underset{\rightarrow}{\hat{\alpha}}| = |\hat{\alpha}|$ and $|\underset{\leftarrow}{\hat{\beta}}| = |\hat{\beta}|$. (See definitions (A.14) and (A.15)). The factor $\sqrt{\underset{\rightarrow}{\hat{\alpha}}!\underset{\leftarrow}{\hat{\beta}}!}$ in this bound will be controlled by the factor $\sqrt{\hat{\alpha}!\hat{\beta}!}$ appearing in the denominator in (A.13). We note the relevant estimate

$$\begin{aligned}\frac{\underset{\rightarrow}{\hat{\alpha}}!}{\hat{\alpha}!} = \frac{\underset{\rightarrow}{\hat{\alpha}}^+!\,\underset{\rightarrow}{\hat{\alpha}}^-!}{\hat{\alpha}^+!\,\hat{\alpha}^-!} &= \prod_{i=1}^M \frac{(\sum_{1<j\leqslant M, j>i}\alpha_{i,j}^+)!}{(\prod_{1<j\leqslant M, j>i}\alpha_{i,j}^+)!}\frac{(\sum_{1<j\leqslant M, j>i}\alpha_{i,j}^-)!}{(\prod_{1<j\leqslant M, j>i}\alpha_{i,j}^-)!} \\ &\leqslant M^{\sum_{1\leqslant i<j\leqslant M}(|\alpha_{i,j}^+|+|\alpha_{i,j}^-|)} = M^{|\hat{\alpha}|},\end{aligned} \tag{B.10}$$

where we made use of properties of the multinomial coefficients. Similarly, the factor $\sqrt{(\underline{\mu}+\underline{\nu})!}$ appearing in (B.9) will be counterbalanced by $\sqrt{\underline{\mu}!\underline{\nu}!}$ extracted from the denominator of (A.13). The relevant estimate relies on the property of the binomial coefficients

$$\frac{(\underline{\mu}+\underline{\nu})!}{\underline{\mu}!\underline{\nu}!} \leqslant 2^{|\underline{\mu}|+|\underline{\nu}|}. \tag{B.11}$$

With the help of the last two bounds and relations (A.13) and (B.9) we obtain

$$\sum_{\substack{\underline{\mu},\underline{\nu}\\ \hat{\alpha},\hat{\beta}\\ (|\hat{\alpha}|,|\hat{\beta}|)\neq(0,0)}} \|\tau_{\underline{\mu}+\underline{\nu}+\overrightarrow{\hat{\alpha}}+\overleftarrow{\hat{\beta}}}\|\|S_{\underline{\mu},\underline{\nu},\hat{\alpha},\hat{\beta}}\| \leqslant \sum_{\underline{\mu},\underline{\nu}}\left(\frac{(4\sqrt{6M_E})^{|\underline{\mu}|+|\underline{\nu}|}}{\sqrt{\underline{\mu}!\underline{\nu}!}}t^{\underline{\mu}+\underline{\nu}}\right)$$
$$\cdot \sum_{\substack{\hat{\alpha},\hat{\beta}\\ (|\hat{\alpha}|,|\hat{\beta}|)\neq(0,0)}}\left(\sqrt{\frac{|\hat{\alpha}^+|!|\hat{\alpha}^-|!|\hat{\beta}^+|!|\hat{\beta}^-|!}{\hat{\alpha}!\hat{\beta}!}}(4\sqrt{3M}g(\delta))^{|\hat{\alpha}|+|\hat{\beta}|}t^{\hat{\alpha}+\hat{\beta}}\right), \tag{B.12}$$

where we made use of the bound (B.10). The sum w.r.t. $\underline{\mu}$, $\underline{\nu}$ can be easily estimated as it factorizes into $4M$ independent sums: let μ be an ordinary multi-index, then

$$\sum_{\underline{\mu},\underline{\nu}}\left(\frac{(4\sqrt{6M_E})^{|\underline{\mu}|+|\underline{\nu}|}}{\sqrt{\underline{\mu}!\underline{\nu}!}}t^{\underline{\mu}+\underline{\nu}}\right) = \left(\sum_{\mu}\frac{(4\sqrt{6M_E})^{|\mu|}}{\sqrt{\mu!}}t^{\mu}\right)^{4M}$$
$$\leqslant \left(\sum_{k=0}^{\infty}\frac{(4\sqrt{6M_E})^{k}}{\sqrt{k!}}\sum_{\mu,|\mu|=k}\frac{|\mu|!}{\mu!}t^{\mu}\right)^{4M}$$
$$\leqslant \left(\sum_{k=0}^{\infty}\frac{(4\sqrt{6M_E}\|T\|_1)^{k}}{\sqrt{k!}}\right)^{4M}, \tag{B.13}$$

where in the second step we made use of the fact that the multinomial coefficients are greater than or equal to one and in the last step we used the multinomial formula. Clearly, the last sum is convergent. (As a matter of fact it would suffice to consider $k \leqslant M_E$ since $S_{\underline{\mu},\underline{\nu},\hat{\alpha},\hat{\beta}}$, given by formula (A.12), vanishes for $|\underline{\mu}| > M_E$ or $|\underline{\nu}| > M_E$.) As for the sum w.r.t. $\hat{\alpha}$, $\hat{\beta}$ on the rhs of (B.12), it suffices to study the case $|\hat{\alpha}^+| \neq 0$. Then the sum factorizes into four independent sums and we discuss here one of the factors

$$\sum_{\hat{\alpha}^+,|\hat{\alpha}^+|\neq 0}\left(\sqrt{\frac{|\hat{\alpha}^+|!}{\hat{\alpha}^+!}}\right)(4\sqrt{3M}g(\delta))^{|\hat{\alpha}^+|}t^{\hat{\alpha}^+}$$
$$= \sum_{\hat{\alpha}^+,|\hat{\alpha}^+|\neq 0}(4\sqrt{3M}g(\delta))^{|\hat{\alpha}^+|}\left(\sqrt{\frac{(|\alpha^+_{1,2}|+\cdots+|\alpha^+_{M-1,M}|)!}{\alpha^+_{1,2}!\cdots\alpha^+_{M-1,M}!}}\right)t^{\hat{\alpha}^+}$$
$$\leqslant \sum_{\hat{\alpha}^+,|\hat{\alpha}^+|\neq 0}(4\sqrt{3M^3}g(\delta))^{|\hat{\alpha}^+|}\frac{|\alpha^+_{1,2}|!}{\alpha^+_{1,2}!}\cdots\frac{|\alpha^+_{M-1,M}|!}{\alpha^+_{M-1,M}!}t^{\hat{\alpha}^+}$$
$$\leqslant \sum_{\substack{k^+_{1,2},\ldots,k^+_{M-1,M}\\ \sum_{1\leqslant i<j\leqslant M}k^+_{i,j}\neq 0}}\prod_{1\leqslant i<j\leqslant M}(4\sqrt{3M^3}g(\delta))^{k^+_{i,j}}\sum_{\alpha^+_{i,j},|\alpha^+_{i,j}|=k^+_{i,j}}\frac{|\alpha^+_{i,j}|!}{\alpha^+_{i,j}!}t^{\alpha^+_{i,j}}$$
$$\leqslant \sum_{\substack{k^+_{1,2},\ldots,k^+_{M-1,M}\\ \sum_{1\leqslant i<j\leqslant M}k^+_{i,j}\neq 0}}(4\sqrt{3M^3}g(\delta)\|T\|_1)^{(k^+_{1,2}+\cdots+k^+_{M-1,M})}. \tag{B.14}$$

In the second step we made use of the fact that

$$\frac{(|\alpha^+_{1,2}|+\cdots+|\alpha^+_{M-1,M}|)!}{|\alpha^+_{1,2}|!\cdots|\alpha^+_{M-1,M}|!} \leqslant M^{2(|\alpha^+_{1,2}|+\cdots+|\alpha^+_{M-1,M}|)} \tag{B.15}$$

and in the last step we exploited the multinomial formula. The last expression on the rhs of (B.14) is a convergent geometric series for sufficiently large δ and it tends to zero with $\delta \to \infty$, since $\lim_{\delta\to\infty} g(\delta) = 0$.

References

[1] Borchers H J 1965 On the vacuum state in quantum field theory: II *Commun. Math. Phys.* **1** 57–79
[2] Borchers H J, Haag R and Schroer B 1963 The vacuum state in quantum field theory *Nuovo Cimento* **29** 148–62
[3] Araki H and Haag R 1967 Collision cross sections in terms of local observables *Commun. Math. Phys.* **4** 77–91
[4] Buchholz D and Fredenhagen K 1982 Locality and the structure of particle states *Commun. Math. Phys.* **84** 1–54
[5] Buchholz D and Wanzenberg R 1992 The realm of the vacuum *Commun. Math. Phys.* **143** 577–89
[6] Dybalski W 2008 A sharpened nuclearity condition and the uniqueness of the vacuum in QFT *Commun. Math. Phys.* **283** 523–42
[7] Dybalski W 2008 A sharpened nuclearity condition for massless fields *Lett. Math. Phys.* **84** 217–30
[8] Haag R 1992 *Local Quantum Physics* (Berlin: Springer)
[9] Dybalski W 2008 Spectral theory of automorphism groups and particle structures in quantum field theory *PhD Thesis* Universität Göttingen (http://webdoc.sub.gwdg.de/diss/2009/dybalski)
[10] Buchholz D and Porrmann M 1990 How small is the phase space in quantum field theory? *Ann. Inst. Henri Poincaré* **52** 237–57
[11] Buchholz D and Yngvason J 1987 Generalized nuclearity conditions and the split property in quantum field theory *Lett. Math. Phys.* **23** 159–67
[12] Sakai S 1971 *C^*-Algebras and W^*-Algebras* (Berlin: Springer)
[13] Buchholz D, Porrmann M and Stein U 1991 Dirac versus Wigner. Towards a universal particle concept in local quantum field theory *Phys. Lett.* B **267** 377–81
[14] Porrmann M 2004 Particle weights and their disintegration: I *Commun. Math. Phys.* **248** 269–304
[15] Porrmann M 2004 Particle weights and their disintegration: II *Commun. Math. Phys.* **248** 305–33
[16] Bostelmann H 2000 Lokale Algebren und Operatorprodukte am Punkt *PhD Thesis* Universität Göttingen (http://webdoc.sub.gwdg.de/diss/2000/bostelmann/)
[17] Bostelmann H 2005 Operator product expansions as a consequence of phase space properties *J. Math. Phys.* **46** 082304–16
[18] Kosaki H 1984 On the continuity of the map $\phi \rightarrow |\phi|$ from the predual of a W^*-algebra *J. Funct. Anal.* **59** 123–31
[19] Buchholz D, D'Antoni C and Longo R 1987 The universal structure of local algebras *Commun. Math. Phys.* **111** 123–35